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ABSTRACT

This is one in a series of SMSG supplementary and enric!-ent pamphlets for high school students. This series is desig.. to make material for the study of topics of special interest to st.. .ts readily accessible in classroom quantity. Topics covered include graphs, slope, distance, mid-point, proof, equations, and circles. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-10

SUPPLEMENTARY and ENRICHMENT SERIES

PLANE COORDINATE GEOMETRY

Edited by Thomas J. Hill

H-G

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

CONTENTS

Section		Page
1.	Introduction.	1
2.	Coordinate Systems in a Plane	2
3.	How to Plot Points on Graph Paper . . .	7
4.	The Slope of a Non-Vertical Line. . . .	10
5.	Parallel and Perpendicular Lines. . . .	17
6.	The Distance Formula.	20
7.	The Mid-Point Formula	24
8.	Proofs of Geometric Theorems.	27
9.	The Graph of a Condition.	32
10.	How to Describe a Line by an Equation .	35
11.	Various Forms of the Equation of a Line	41
12.	The General Form of the Equation of a Line.	43
13.	Intersection of Lines	46
14.	Circles	51
15.	Sets Satisfying Geometric Conditions. .	57

PLANE COORDINATE GEOMETRY

1. Introduction.

Mathematics is the only science in which practically nothing ever has to be thrown away. Of course, mathematicians are people, and being people, they make mistakes. But these mistakes usually are caught pretty quickly. Therefore, when one generation has learned something about mathematics, the next generation can go on to learn some more, without having to stop to correct serious errors in the work that was supposed to have been done already.

One symptom of this situation is the fact that nearly all of the geometry usually studied in high school was known to the ancient Greeks, over two thousand years ago. The first really big step forward in geometry, after the Greeks, was in the seventeenth century. This was the discovery of a new method, called <u>coordinate geometry</u>, by Rene Descartes (1596-1650).

This pamphlet is intended to give you an introduction to coordinate geometry of the plane--just enough to give you an idea of what it is like and how it works. Coordinate geometry may be described roughly as the application of algebraic principles to the study of geometric relationships. Therefore, it is assumed that you have completed the major part of a year's study of geometry. You should, for example, be familiar with such ideas as congruence, similarity, parallelism, and perpendicularity.

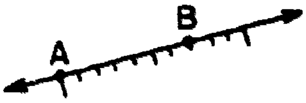
Some of the symbols used in this publication may differ from those you have used. However, they are easily understood; an explanation of some of them is given below.

$\overleftrightarrow{AB}$ represents the <u>line</u> containing the two distinct points A and B. A line has no end-points, but extends indefinitely in both of its senses of direction.

$\overrightarrow{AB}$ represents the <u>ray</u> having A as its end-point, and containing point B. Note that a ray has exactly one end-point.

$\overline{AB}$ represents the segment having A and B as end-points. A segment then has two end-points, and may be described as a set of points consisting of two distinct points of a line together with all points of the line between the two points.

AB represents the distance between points A and B. For example, the figure at the right, suggesting a ruler placed along the representation of line AB, indicates $AB = 7$.



> It is to be noted then that symbols such as $\overleftrightarrow{AB}$, $\overrightarrow{AB}$, and $\overline{AB}$ denote sets of points, or geometric figures. (Any geometric figure is a set, or collection, of points.) On the other hand, a symbol such as AB represents a real number; that is, a measure of distance. (These symbols are purely arbitrary, and the fact that they may be new to you should in no way affect your understanding of geometry.)

$a < b$ means "the number a is less than the number b".
For example, "$3 < 4$" is a true statement; "$4 < 3$" is false.
If $x < 0$, then x is negative.
If $x \leq 0$, then x is zero or negative.

$b > a$ means "the number b is greater than the number a".
In other words, it has the same meaning as "$a < b$".
If $x > 0$, then x is positive.
If $x \geq 0$, then x is zero or positive.

$|a|$ means "absolute value of the number a".
If $a = 0$, $|a| = 0$. That is, $|0| = 0$.
If $a > 0$, $|a| = a$. For example, $|5| = 5$.
If $a < 0$, $|a| = -a$. For example, $|-5| = 5$.
Note that $|a|$ is always a non-negative number. That is, $|a| \geq 0$. Therefore, distances are often expressed in terms of absolute values.

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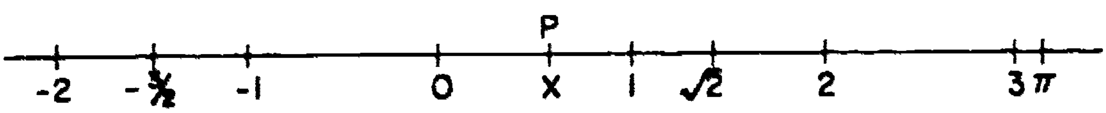
2. Coordinate Systems in a Plane.

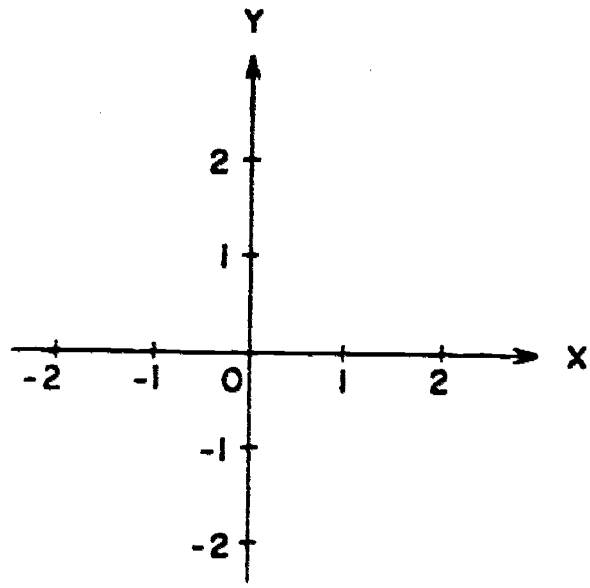


A coordinate system can be set up on a line so that every number describes a point, and every point P on the line is determined

7

when its coordinate x is named. On the line represented above, a coordinate system has been set up, and several coordinates are indicated in the diagram.

In coordinate geometry, we do the same sort of thing in a plane, except that in a plane a point is described not by a single number, but by a pair of numbers. The scheme works like this:



First we take a line X in the plane, and set up a coordinate system on X. This line will be called the x-axis. In a figure we often use an arrowhead to emphasize the positive direction on the x-axis.

Next we let Y be the perpendicular to the x-axis through the point O whose coordinate is zero, and we set up a coordinate system on Y. By the Ruler Placement Postulate* this can be done so that point O also has coordinate zero on Y. Y will be called the y-axis. As before, we indicate the positive direction by an arrowhead. The intersection O of the two axes is called the origin.

We can now describe any point in the plane by a pair of numbers. The scheme is this. Given a point P, we drop a perpendicular to the x-axis, ending at point M, with coordinate x. We drop a perpendicular to the y-axis, ending at point N, with coordinate y. (We can call M and N the projections of P into X and Y.)

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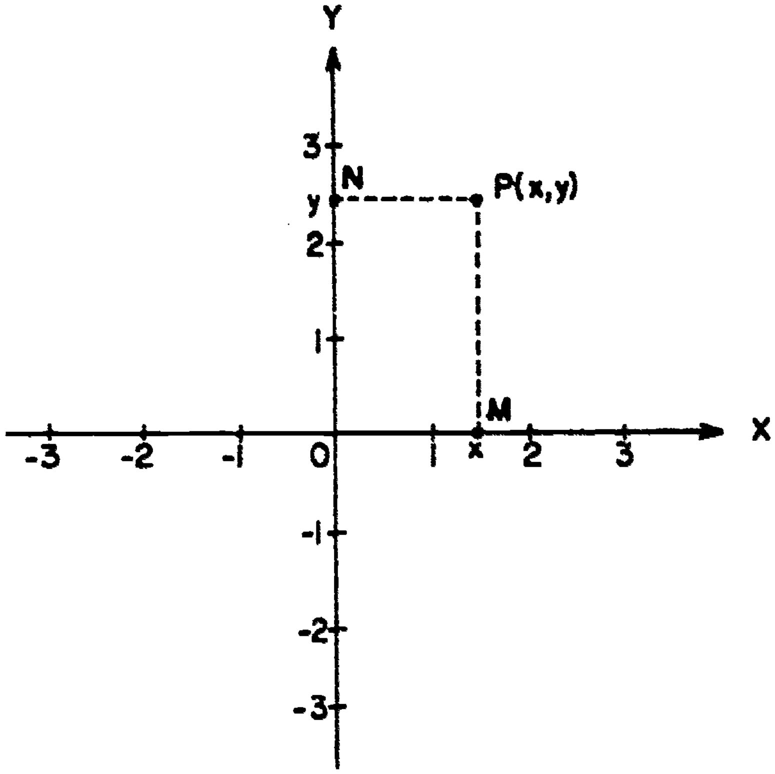
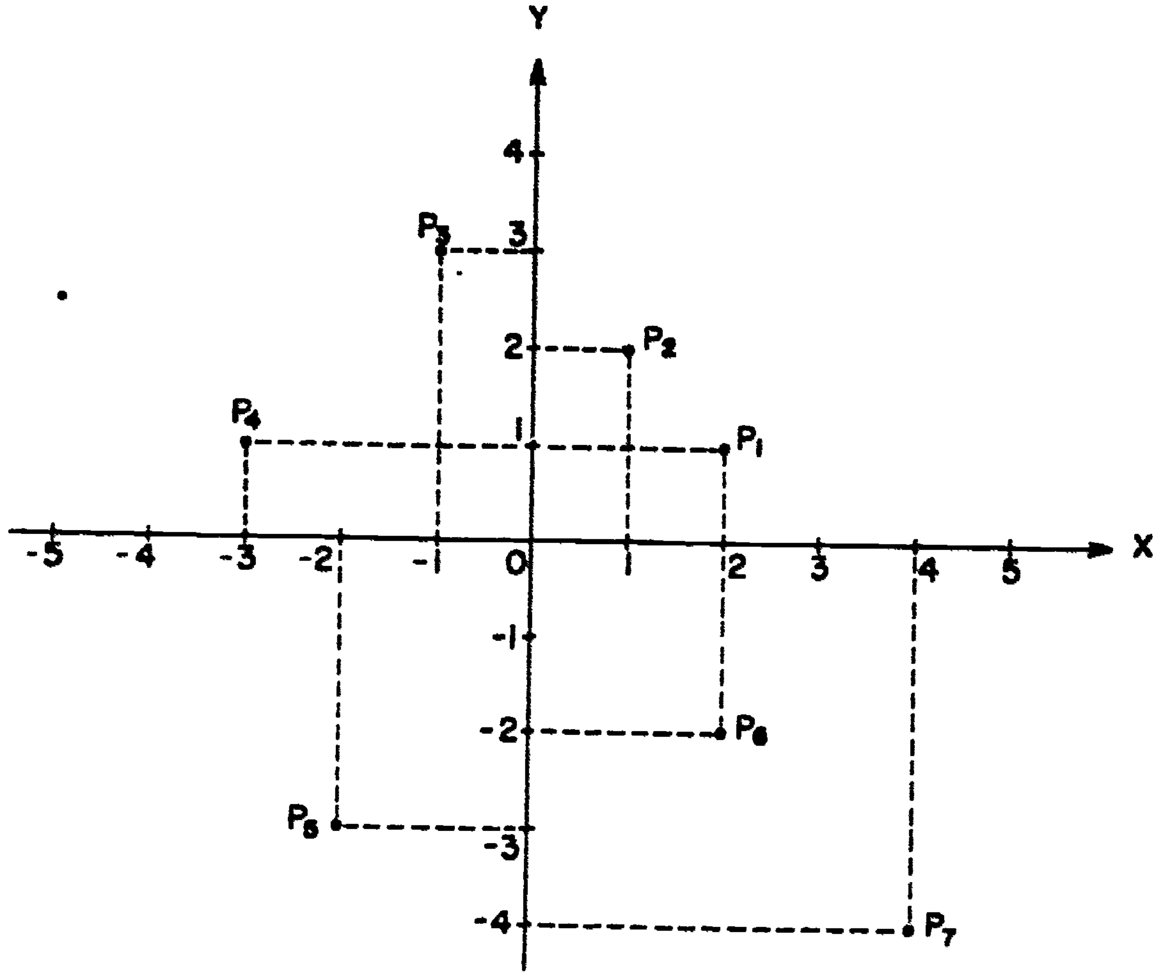
*The Ruler Placement Postulate may be stated in the following way: "Given any two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive."

Definitions: The numbers x and y are called the <u>coordinates</u> of the point P; x is the <u>x-coordinate</u> and y is the <u>y-coordinate</u>.



In the figure $x = 1\frac{1}{2}$ and $y = 2\frac{1}{2}$. The point P therefore has coordinates $1\frac{1}{2}$ and $2\frac{1}{2}$. We write these coordinates in the form $(1\frac{1}{2}, 2\frac{1}{2})$, giving the x-coordinate first. To indicate that point P has these coordinates, we write $P(1\frac{1}{2}, 2\frac{1}{2})$ or $P:(1\frac{1}{2}, 2\frac{1}{2})$.

Let us look at some more examples.



We read off the coordinates of the points by following the dotted lines. Thus, the coordinates, in each case, are as follows:

$P_1(2,1)$

$P_2(1,2)$

$P_3(-1,3)$

$P_4(-3,1)$

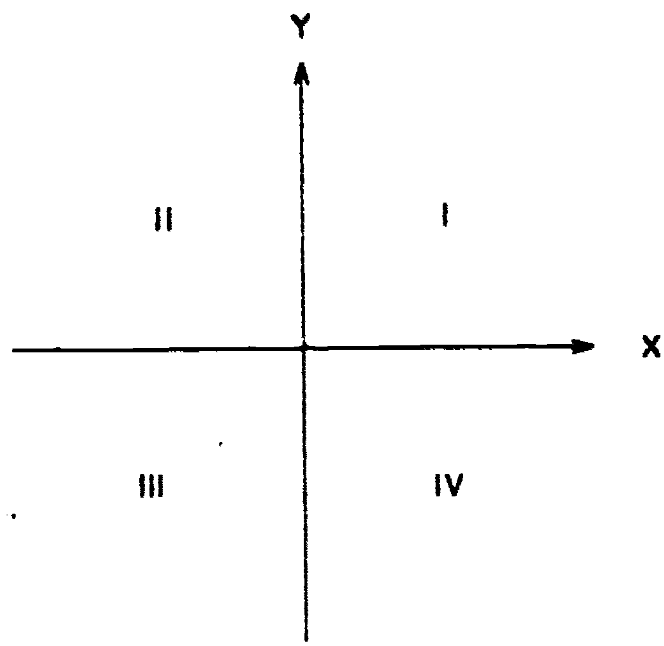
$P_5(-2,-3)$

$P_6(2,-2)$

$P_7(4,-4)$

Notice that the order in which the coordinates are written makes a difference. The point with coordinates (2,1) is not the same as point (1,2). Thus, the coordinates of a point are really an ordered pair of real numbers. The convention of having the first number of the ordered pair be the x-coordinate, and the second the y-coordinate, is highly important.

Just as a single line separates the plane into two parts (called half-planes), so the two axes separate the plane into four parts, called quadrants. The quadrants are identified by number, like this:



We have shown that any point of our plane determines an ordered pair of numbers. Can we reverse the process? That is, given a pair of numbers (a,b) can we find a point whose coordinates are (a,b)? The answer is easily seen to be "yes". In fact, there is exactly one such point, obtained as the intersection of the line perpendicular to the x-axis at the point whose coordinate is a and the line perpendicular to the y-axis at the point whose coordinate is b.

Thus, we have a one-to-one correspondence between points in the plane and ordered pairs of numbers. Such a correspondence is called a coordinate system in the plane. A coordinate system is specified by choosing a measure of distance, an x-axis, a y-axis perpendicular to it and a positive direction on each. As long as we stick to a specific coordinate system, each point P is associated with exactly one number pair (a,b), and each number pair with exactly one point. Hence, it will cause no confusion if we say the number pair is the point, thus enabling us to use such convenient phrases as "the point (2,3)" or "P = (a,b)".

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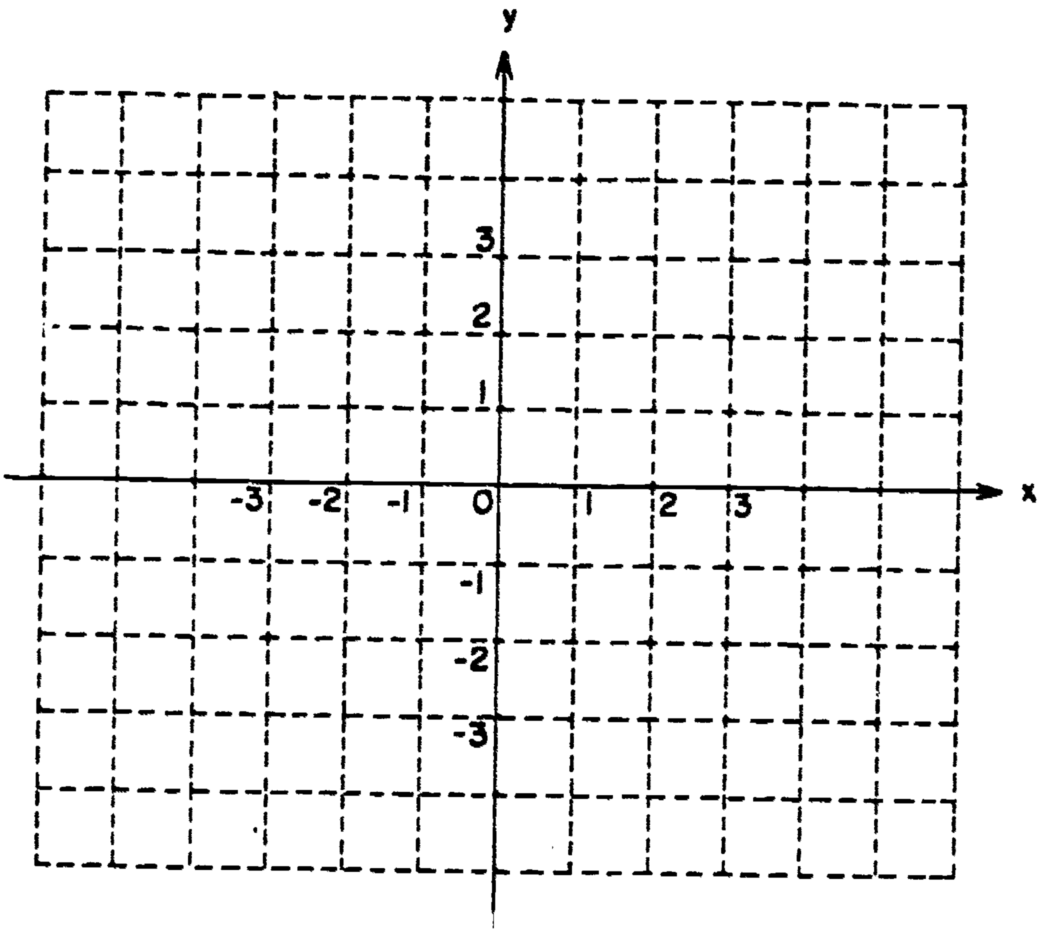
3. How to Plot Points on Graph Paper.

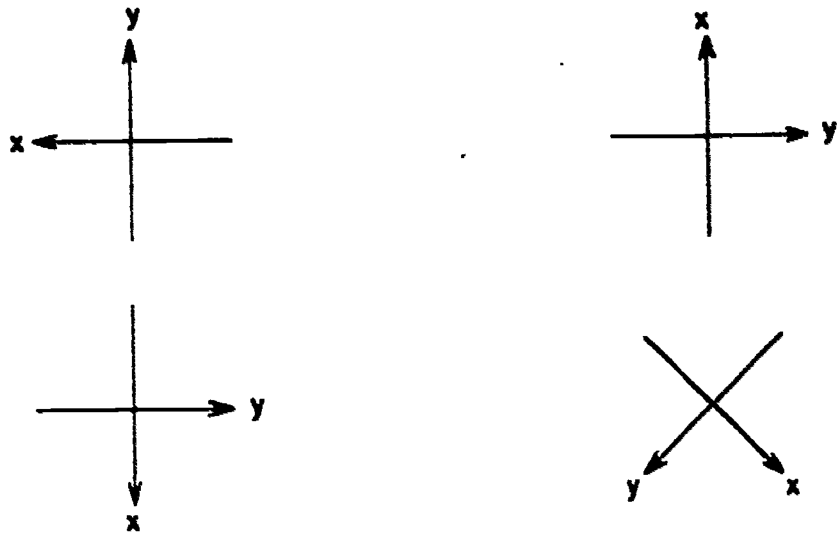
As a matter of convenience, we ordinarily use printed graph paper for drawing figures in coordinate geometry. The horizontal and vertical lines are printed; we have to draw everything else for ourselves.



In the figure above, the dotted lines represent the lines that are already printed on the paper. The x-axis and the y-axis should be drawn with a pen or a pencil. Notice that the x-axis is labeled x rather than X; this is customary. Here the symbol x is not the name of anything, but merely a reminder that the coordinates on this axis are going to be denoted by the letter x. Similarly, for the y-axis. Next, the points with coordinates (1,0) and (0,1) must be labeled in order to indicate the unit to be used.

This is the usual way of preparing graph paper for plotting points. We could have indicated a little less or a lot more. For your own convenience, it is a good idea to show more than this. But if you show less, then your work may be actually unintelligible.

Note that we could draw the axes in any of the following positions:



and so on. There is nothing logically wrong with any of these ways of drawing the axes. People find it easier to read each other's graphs, however, if they agree at the outset that the x-axis is to be horizontal, with coordinates increasing from left to right, and the y-axis is to be vertical, with coordinates increasing from bottom to top.

Problem Set 1.

1. Suggest why the kind of coordinate system used in this pamphlet is sometimes called "Cartesian".

2. What are the coordinates of the origin?

3. What is the y-coordinate of the point (7,-3)?

4. Name the point which is the projection of (0,-4) into the x-axis.

5. Which pair of points are closer together, (2,1) and (1,2) or (2,1) and (2,0)?

6. In which quadrant is each of the following points?

a. (5,-3).
b. (-5,3).
c. (5,3).
d. (-5,-3).

7. What are the coordinates of a point which does not lie in any quadrant?

8. The following points are projected into the x-axis. Write them in such an order that their projections will be in order from left to right.

A:(6,-3). B:(-2,5). C:(0,-4). D:(-5,).

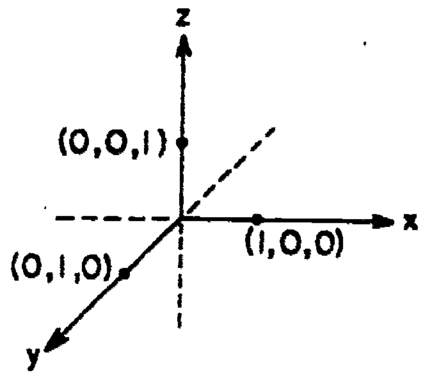
9. If the points in the previous problem are projected into the y-axis arrange them so their projections will be in order from bottom to top.

10. If s is a negative number and r a positive number, in what quadrant will each of the following points lie?

a. (s,r).
b. (-s,r).
c. (-s,-r).
d. (s,-r).
e. (r,s).
f. (r,-s).
g. (-r,-s).
h. (-r,s).

11. Set up a coordinate system on graph paper. Using segments, draw some simple picture on the paper. On a separate paper list in pairs the coordinates of the endpoints of the segments in your picture. Exchange your list of coordinates with another student, and reproduce the picture suggested by his list of coordinates.

*12. A three-dimensional coordinate system can be formed by considering three mutually perpendicular axes as shown. The y-axis, while drawn on this paper, represents a line perpendicular to the plane of the paper. The negative portions of the x, y and z axes extend to the left, to the rear, and down, respectively. Taken in pairs, the three axes determine three planes called the yz-plane, the xz-plane, and the xy-plane. A point (x,y,z) is located by its three coordinates: the x-coordinate



is the coordinate of its projection into the x-axis; the y and z coordinates are defined in a corresponding manner.

a. On which axis will each of these points lie?

$(0,5,0)$; $(-1,0,0)$; $(0,0,8)$.

b. On which plane will each of these points lie?

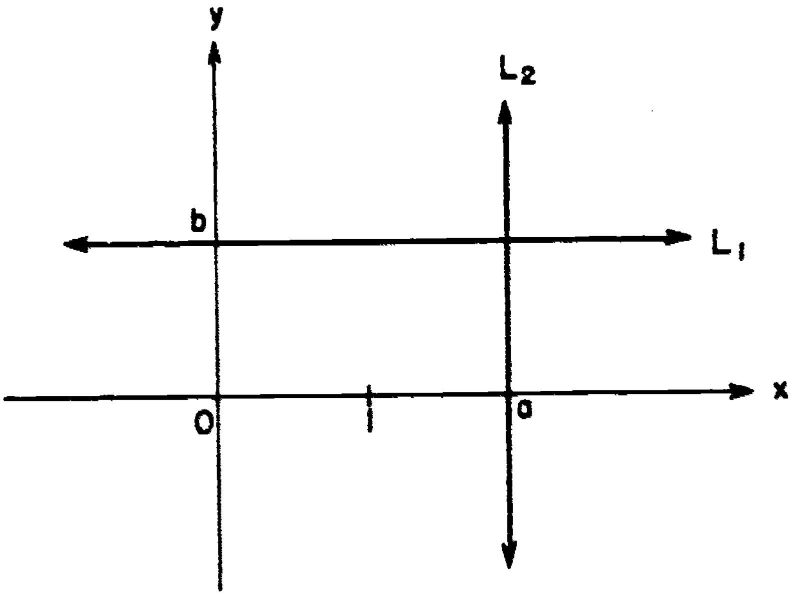
$(2,0,3)$; $(0,5,-7)$; $(1,1,0)$.

c. What is the distance of the point $(3,-2,4)$ from the xy-plane? from the xz-plane? from the yz-plane?

---

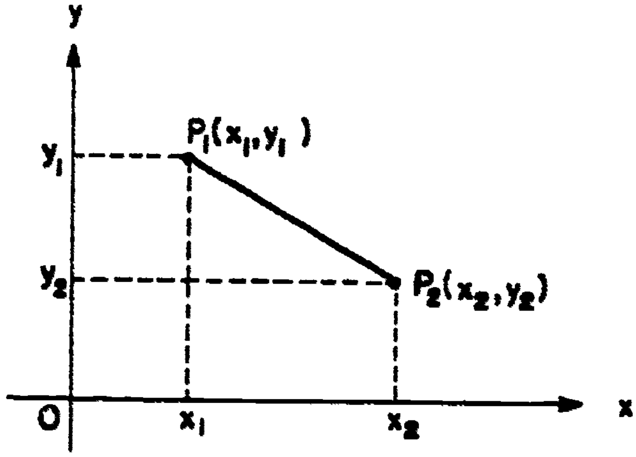
4. The Slope of a Non-Vertical Line.

The x-axis and all lines parallel to it are called horizontal. The y-axis and all lines parallel to it are called vertical. Notice that these words are defined in terms of the coordinate system that we have set up.



On the horizontal line L_1, all points have the same y-coordinate b, because the point $(0,b)$ on the y-axis is the foot of all the perpendiculars from points of L_1. For the same sort of reason, all points of the vertical line L_2 have the same x-coordinate a. Of course, a segment is horizontal (or vertical) if the line containing it is horizontal (or vertical).

Consider now a segment $\overline{P_1P_2}$, where $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$, and suppose that $\overline{P_1P_2}$ is not vertical.



<u>Definition</u>: The slope of $\overline{P_1P_2}$ is the number $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

This really is a number: since the segment is not vertical, P_1 and P_2 have different x-coordinates, and so the denominator is not zero. Some things about the slope are easy to see.

(1) It is important that the order of naming the coordinates is the same in the numerator as in the denominator. Thus, if we wish to find the slope of $\overline{PQ}$, where $P = (1,3)$ and $Q = (4,2)$ we can either choose $P_1 = P$, $x_1 = 1$, $y_1 = 3$, $P_2 = Q$, $x_2 = 4$, $y_2 = 2$, giving slope of $\overline{PQ} = \frac{2-3}{4-1} = -\frac{1}{3}$;

or $P_1 = Q$, $x_1 = 4$, $y_1 = 2$, $P_2 = P$, $x_2 = 1$, $y_2 = 3$, giving slope of $\overline{PQ} = \frac{3-2}{1-4} = -\frac{1}{3}$.

What we <u>cannot</u> say is

$$\text{slope of } \overline{PQ} = \frac{3-2}{4-1} \text{ or } \frac{2-3}{1-4}.$$

Notice that if the points are named in reverse order, the slope is the same as before. Algebraically,

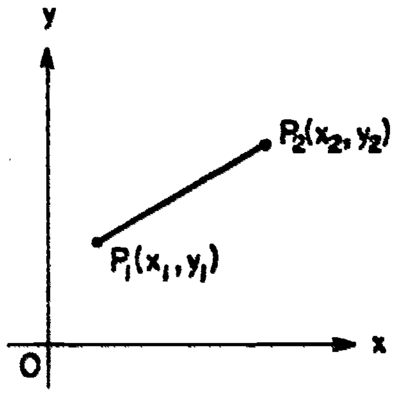
$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}.$$

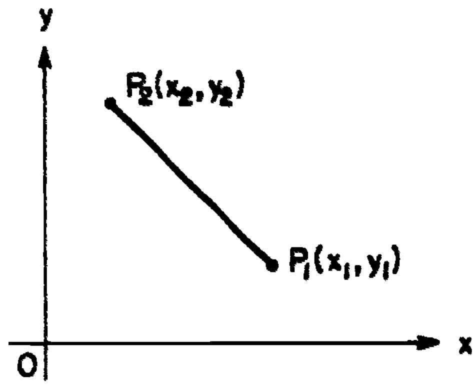
Thus, the value of m depends only on the segment, not on the order in which the endpoints are named.

(2) If $m = 0$, then the segment is horizontal. (Algebraically, a fraction is zero only if its numerator is zero, and this means that $y_2 = y_1$.)

(3) If the segment slopes upward from left to right, as in the left-hand figure below, then $m > 0$, because the numerator and denominator are both positive (or both negative, if we reverse the order of the endpoints).

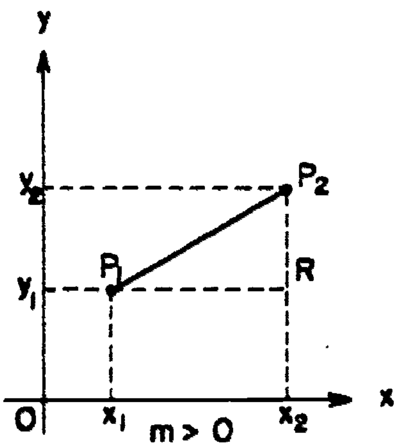
(4) If the segment slopes upward from right to left as in the right-hand figure below, then $m < 0$. This is because m can be written as a fraction with a positive numerator $y_2 - y_1$ and a negative denominator $x_2 - x_1$ (or equivalently, a negative numerator $y_1 - y_2$ and a positive denominator $x_1 - x_2$).

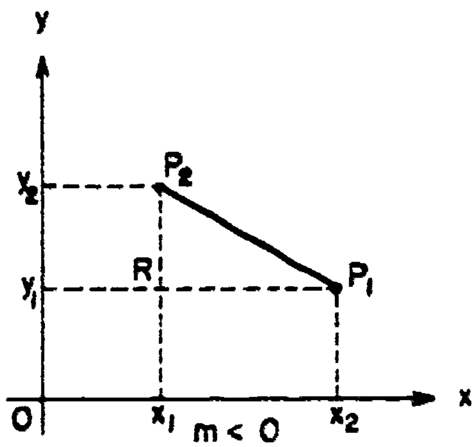




(5) We do not try to write the slope of a vertical segment, because the denominator would be zero, and so the fraction would be meaningless.

In either of the two figures above, we can complete a right triangle $\triangle P_1P_2R$, by drawing horizontal and vertical lines through P_1 and P_2, like this:





Since opposite sides of a rectangle are congruent, it is easy to see that

(1) If $m > 0$, then $m = \dfrac{RP_2}{P_1R}$ and

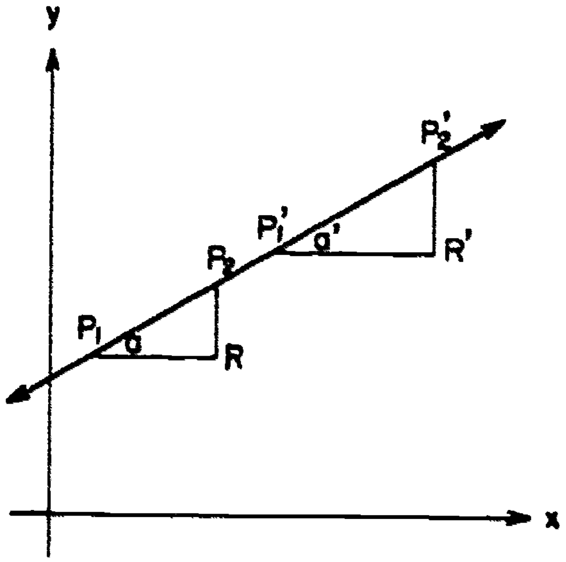
(2) if $m < 0$, then $m = -\frac{RP_2}{P_1R}$.

Once we know this much about slopes, it is easy to get our first basic theorem.

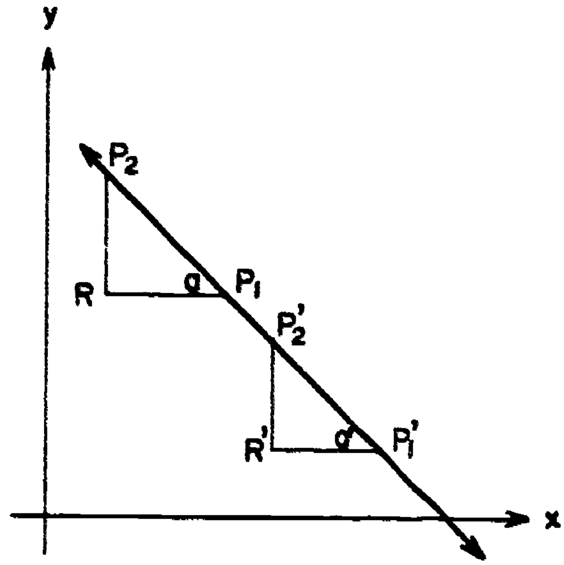
Theorem 1. On a non-vertical line, all segments have the same slope.

Proof: There are three cases to be considered.

Case (1): If the line is horizontal, all segments on it have slope zero.



Case (2)



Case (3)

In either of the cases illustrated above, $\angle a \cong \angle a'$, and since the triangles are right triangles, this means that

$$\triangle P_1P_2R \sim \triangle P_1'P_2'R'.$$

Therefore, in either case,

$$\frac{RP_2}{P_1R} = \frac{R'P_2'}{P_1'R'}.$$

In Case (2), these fractions are the slopes of $\overline{P_1P_2}$ and $\overline{P_1'P_2'}$ and therefore the segments have the same slope. In Case (3) the slopes are the negatives of the same fractions, and are therefore equal.

Theorem 1 means that we can talk not only about the slopes of segments but also about the slopes of lines: the slope of a non-vertical line is the number m which is the slope of every segment of the line.

Problem Set 2

1. Replace the "?" in such a way that the line through the two points will be horizontal.

   (a) (5,7) and (-3,?).
   (b) (0,-1) and (4,?).
   (c) (x_1,y_1) and $(x_2,?)$.

2. Replace the "?" in such a way that the line through the two points will be vertical.

   (a) (?,2) and (6,-4).
   (b) (-3,-1) and (?,0).
   (c) (x_1,y_1) and $(?,y_2)$.

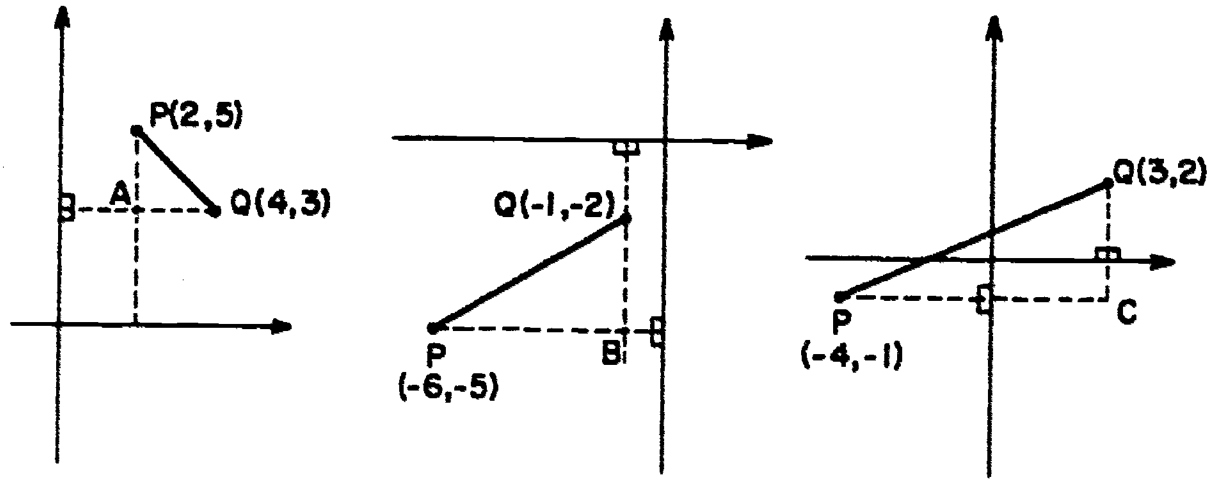
3. By visualizing the points on a coordinate system in parts (a), (b), and (c), give the distance between:

   (a) (5,0) and (7,0).
   (b) (5,1) and (7,1).
   (c) (-3,-4) and (-6,-4).
   (d) What is alike about parts (a), (b) and (c)?
   (e) State a rule giving an easy method for finding the distance between such pairs of points.
   (f) Does your rule apply to the distance betwen (6,5) and (3,-5)?

4. By visualizing the points named in parts (a), (b), (c) and (d) on a coordinate system, give the distance between the points in each part.

   (a) (7,-3) and (7,0).
   (b) (-3,1) and (-3,-1).
   (c) (6,8) and (6,4).
   (d) (x_1,y_1) and (x_1,y_2).
   (e) What is alike about parts (a), (b), (c) and (d)?
   (f) State a rule giving an easy method for finding the distance between such pairs of points.

5. With perpendiculars drawn as shown below, what are the coordinates of A, B and C?



6. Determine the distances from P and Q to points A, B and C in Problem 5.

7. Compute the slope of $\overline{PQ}$ for each figure in Problem 5.

8. A road goes up 2 feet for every 30 feet of horizontal distance. What is its slope?

9. Determine the slope of the segment joining each of the following point pairs.

   (a) (0,0) and (6,2).
   (b) (0,0) and (2,-6).
   (c) (3,5) and (7,12).
   (d) (0,0) and (-4,-3).
   (e) (-5,7) and (3,-8).
   (f) $(\frac{1}{2},\frac{1}{3})$ and $(\frac{1}{4},\frac{1}{5})$.
   (g) (-2.8,3.1) and (2.2,-1.9).
   (h) $(\frac{1}{240},0)$ and $(0,\frac{1}{80})$.

10. Replace the "?" by a number so that the line through the two points will have the slope given. (Hint: Substitute in the slope formula.)

    (a) (5,2) and (?,6). $m = 4$.
    (b) (-3,1) and (4,?). $m = \frac{1}{2}$.

*11. $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are non-vertical lines. Prove that $\overleftrightarrow{PA} = \overleftrightarrow{PB}$ if and only if they have the same slope; and consequently if $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ have different slopes, then P, A and B cannot be collinear. (A set of points is collinear if every point in the set lies on one line.)

12. (a) Is the point B(4,13) on the line joining A(1,1) to C(5,17)? (Hint: Is the slope of $\overline{AB}$ the same as that of $\overline{BC}$?)

    (b) Is the point (2,-1) on the segment joining (-5,4) o (6,-8)?

13. Determine the slope of a segment joining:
    (a) (0,n) and (n,0).
    (b) (2d,-2d) and (0,d).
    (c) (a + b,a) and (a - b,b).

14. Given A:(101,102), B:(5,6), C:(-95,-94), determine whether or not lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ coincide.

15. Given A:(101,102), B:(5,6), C:(202,203), D:(203,204). Are $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ parallel? Could they possibly coincide?

16. Draw the part of the first quadrant of a coordinate system having coordinates less than or equal to 5. Draw a segment through the origin which, if extended, would pass through P(80000000,60000000).

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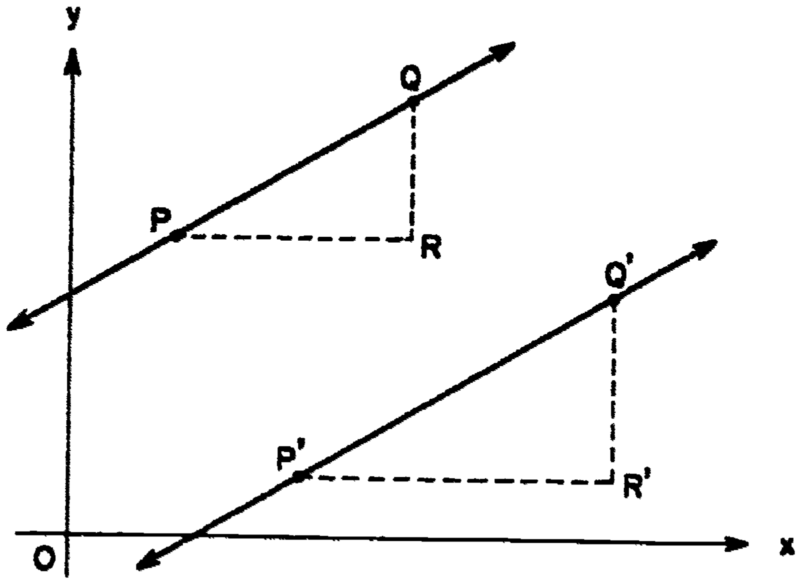
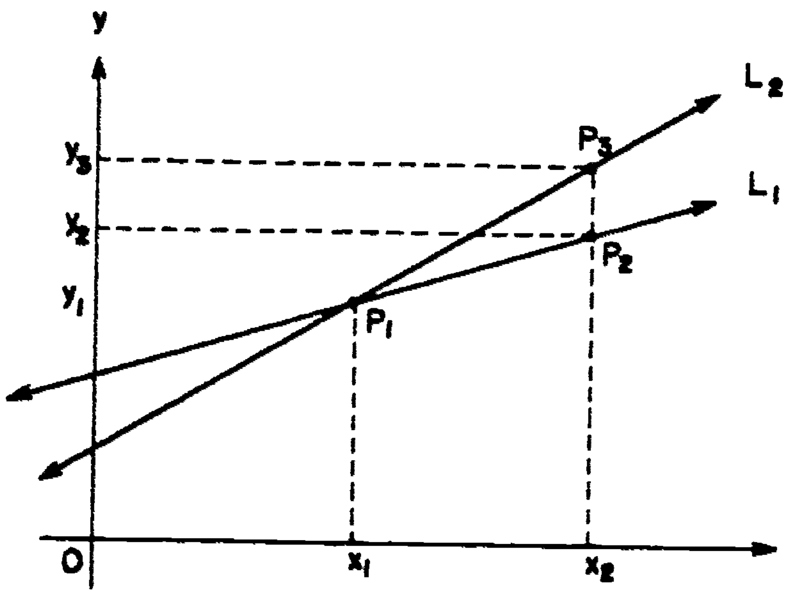
5. Parallel and Perpendicular Lines.

It is easy to see the algebraic condition for two non-vertical lines to be parallel.



If the lines are parallel, then $\triangle PQR \sim \triangle P'Q'R'$, and it follows, as in the proof of the preceding theorem, that they have the same slope.

Conversely, if two different lines have the same slope, then they are parallel. We prove this by the method of contradiction.



Assume as in the figure above that L_1 and L_2 are not parallel. If, as shown in the figure, P_1 is their point of intersection, and P_2 and P_3 have the same x-coordinate x_2, the slope of

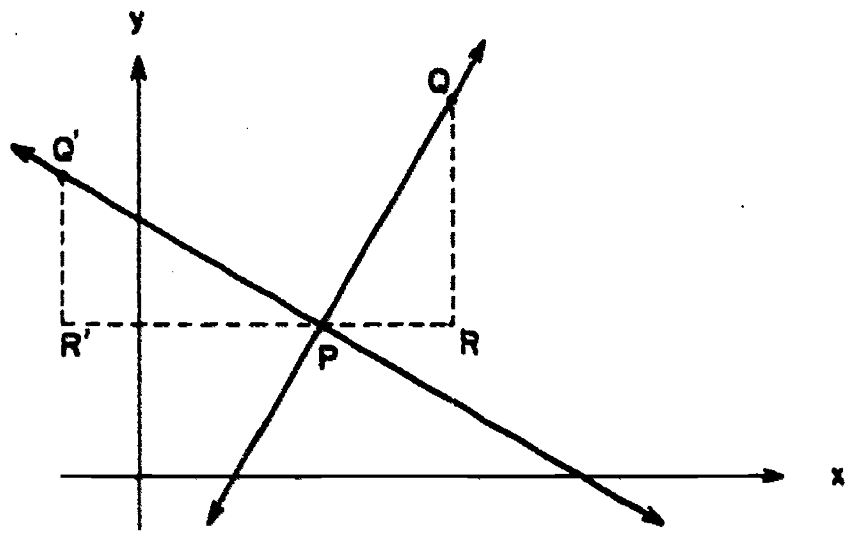
L_1 is $m_1 = \dfrac{y_2 - y_1}{x_2 - x_1}$, and the slope of L_2 is $m_2 = \dfrac{y_3 - y_1}{x_2 - x_1}$.

Since $y_3 \neq y_2$, the fractions cannot be equal, and hence, $m_1 \neq m_2$. Thus, our initial assumption that the two lines L_1 and L_2 were not parallel has led us to a contradiction of the hypothesis that $m_1 = m_2$. Hence, the two lines L_1 and L_2 must be parallel.

Thus, we have the theorem:

<u>Theorem</u> 2. Two non-vertical lines are parallel if and only if they have the same slope.

Now turning to the condition for two lines to be perpendicular, let us suppose that we have given two perpendicular lines, neither of which is vertical.



Let P be their point of intersection. As in the figure, let Q be a point of one of the lines, lying above and to the right of P. And let Q' be a point of the other line, lying above and to the left of P, such that PQ' = PQ. We complete the right triangles $\triangle PQR$ and $\triangle Q'PR'$ as indicated in the figure. Then

$$\triangle PQR \cong \triangle Q'PR'. \qquad \text{(Why?)}$$

Therefore, $\qquad Q'R' = PR$ and $R'P = RQ$,

and, hence, $$\frac{Q'R'}{R'P} = \frac{PR}{RQ}.$$

Let m be the slope of $\overleftrightarrow{PQ}$, and let m' be the slope of $\overleftrightarrow{PQ'}$.

Then $$m = \frac{RQ}{PR},$$

and $$m' = -\frac{1}{m}.$$

That is, the slopes of perpendicular lines are the negative reciprocals of each other.

Suppose, conversely, that we know that $m' = -\frac{1}{m}$. We then construct $\triangle PQR$ as before, and we construct the right triangle $\triangle Q'PR'$ making $R'P = RQ$. We can then prove that $Q'R' = PR$; this gives the same congruence, $\triangle PQR \cong \triangle Q'PR'$ as before, and it follows that $\angle Q'PQ$ is a right angle and, hence, $\overleftrightarrow{PQ} \perp \overleftrightarrow{PQ'}$.

These two facts are stated together in the following theorem:

> Theorem 3. Two non-vertical lines are perpendicular if and only if their slopes are the negative reciprocals of each other.

Notice that while Theorems 2 and 3 tell us nothing about vertical lines, they don't really need to because the whole problem of parallelism and perpendicularity is trivial when one of the lines is vertical. If L is vertical, then L' is parallel to L if and only if L' is also vertical (and different from L.) And if L is vertical, then L' is perpendicular to L if and only if L' is horizontal.

Problem Set 3.

1. Four points taken in pairs determine six segments. Which pairs of segments determined by the following four points are parallel? $A(3,6)$; $B(5,9)$; $C(8,2)$; $D(6,-1)$. (Caution: Two segments are not necessarily parallel if they have the same slope!)

2. Show by considering slopes that a parallelogram is formed by drawing segments joining, in order, $A(-1,5)$, $B(5,1)$, $C(6,-2)$ and $D(0,2)$.

3. Lines L_1, L_2, L_3 and L_4 have slopes $\frac{2}{3}$, -4, $-1\frac{1}{2}$ and $\frac{1}{4}$. respectively. Which pairs of lines are perpendicular?

4. It is asserted that both of the quadrilaterals whose vertices are given below are parallelograms. Without plotting the points, determine whether or not this is true.

   (1) $A:(-5,-2)$ $B:(-4,2)$, $C:(4,6)$, $D:(3,1)$.

   (2) $P:(-2,-2)$ $Q:(4,2)$, $R:(9,1)$, $S:(3,-3)$.

5. The vertices of a triangle are $A(16,0)$, $B(9,2)$ and $C(0,0)$.

   (a) What are the slopes of its sides?

   (b) What are the slopes of its altitudes?

6. Show that the quadrilateral joining $A(-2,2)$, $B(2,-2)$, $C(4,2)$, and $D(2,4)$ is a trapezoid with perpendicular diagonals.

7. Show that a line through $(3n,0)$ and $(0,n)$ is parallel to a line through $(6n,0)$ and $(0,2n)$.

8. Show that a line through $(0,0)$ and (a,b) is perpendicular to a line through $(0,0)$ and $(-b,a)$.

*9. Show that if a triangle has vertices $X(r,s)$, $Y(na+r,nb+s)$ and $Z(-mb+r,ma+s)$ it will have a right angle at X.

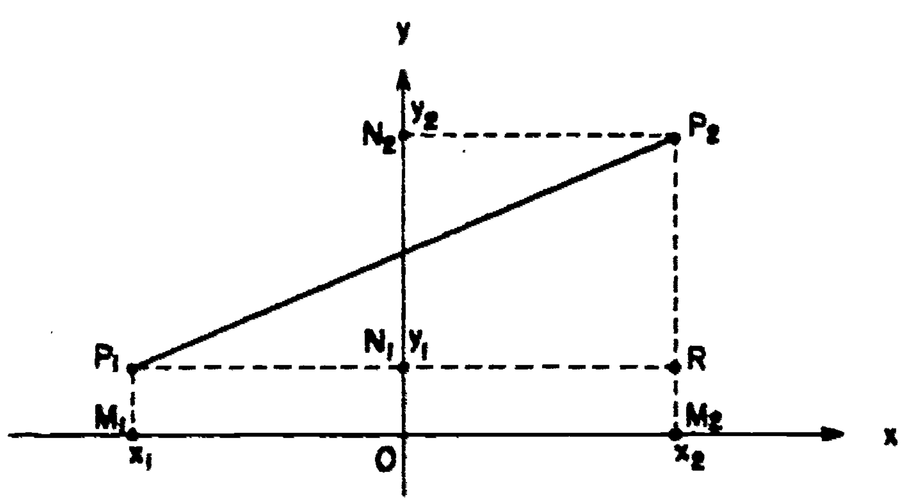
10. Given the points $P(1,2)$, $Q(5,-6)$ and $R(b,b)$; determine the value of b so that $\angle PQR$ is a right angle.

11. $P = (a,1)$, $Q = (3,2)$, $R = (b,1)$, $S = (4,2)$. Prove that $\overleftrightarrow{PQ} \nparallel \overleftrightarrow{RS}$, and that if $\overline{PQ} \parallel \overline{RS}$ then $a = b - 1$.

---

6. The Distance Formula.

If we know the coordinates of two points P_1 and P_2, then we know where the points are, and so the distance P_1P_2 is determined. Let us now find out how the distance can be calculated. What we want is a formula that gives P_1P_2 in terms of the coordinates x_1, x_2, y_1 and y_2.



Let the projections M_1, M_2, N_1 and N_2 be as in the figure. By the Pythagorean Theorem, $(P_1P_2)^2 = (P_1R)^2 + (RP_2)^2$.

Also $$P_1R = M_1M_2 \text{ and } RP_2 = N_1N_2,$$

because opposite sides of a rectangle are congruent.

Therefore, $$(P_1P_2)^2 = (M_1M_2)^2 + (N_1N_2)^2.$$

But we know that $$M_1M_2 = |x_2 - x_1|$$

and $$N_1N_2 = |y_2 - y_1|.$$

Therefore, $$(P_1P_2)^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 .$$

Of course, the square of the absolute value of a number is the same as the square of the number itself.

Therefore, $$(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

and since $P_1P_2 \geq 0$, this means that

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$

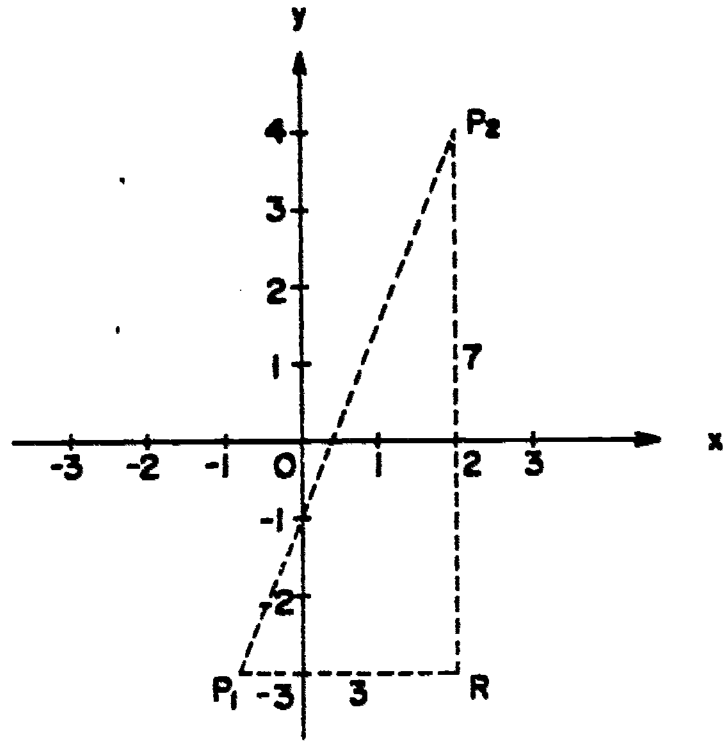
This is the formula that we are looking for. Thus, we have the theorem:

Theorem 4. (The Distance Formula.) The distance between the points (x_1,y_1) and (x_2,y_2) is equal to

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$

For example, take $P_1 = (-1,-3)$ and $P_2 = (2,4)$.

By formula, $$\begin{aligned} P_1P_2 &= \sqrt{(2 + 1)^2 + (4 + 3)^2} \\ &= \sqrt{9 + 49} \\ &= \sqrt{58} \end{aligned}$$



Of course, if we plot the points, as above, we can get the same answer directly from the Pythagorean Theorem; the legs of the right triangle $\triangle P_1RP_2$ have lengths 3 and 7, so that $P_1P_2 = \sqrt{3^2 + 7^2}$, as before. If we find the distance this way, we are of course simply repeating the derivation of the distance formula in a specific case.

Problem Set 4.

1. (a) <u>Without</u> using the distance formula, state the distance between each pair of the points: A(0,3), B(1,3), C(-3,3) and D(4.5,3).

   (b) <u>Without</u> using the distance formula, state the distance between each pair of the points: A(2,0), B(2,1), C(2,-3) and D(2,4.5).

2. (a) Write a simple formula for the distance between (x_1,k) and (x_2,k). (Hint: The points lie on a horizontal line.)

   (b) Write a simple formula for the distance between (k,y_1) and (k,y_2).

3. Use the distance formula to find the distance between:

(a) (0,0) and (3,4).
(b) (0,0) and (3,-4).
(c) (1,2) and (6,14).
(d) (8,11) and (15,35).
(e) (3,8) and (-5,-7).
(f) (-2,3) and (-1,4).
(g) (10,1) and (49,81).
(h) (-6,3) and (4,-2).

4. (a) Write a formula for the square of the distance between the points (x_1,y_1) and (x_2,y_2).

   (b) Using coordinates, write and simplify the statement:
   The square of the distance between (0,0) and (x,y) is 25.

5. Show that the triangle with vertices R(0,0), S(3,4) and T(-1,1) is isosceles by computing the lengths of its sides.

6. Using the converse of the Pythagorean Theorem, show that the triangle joining D(1,1), E(3,0) and F(4,7) is a right triangle with a right angle at D.

7. Given the points A(-1,6), B(1,4) and C(7,-2). Prove, <u>without</u> plotting the points, that B is between A and C.

8. Suppose the streets in a city form congruent square blocks with avenues running east-west and streets north-south.

   (a) If you follow the sidewalks, how far would you have to walk from the corner of 4th Avenue and 8th Street to the corner of 7th Avenue and 12th Street? (Use the length of 1 block as your unit of length.)

   (b) What would be the distance "as the crow flies" between the same two corners?

9. Vertices W, X and Z of rectangle WXYZ have coordinates (0,0), (a,0) and (0,b), respectively.

   (a) What are the coordinates of Y?

   (b) Prove, using coordinates, that WY = XZ.

*10. (a) Using 3-dimensional coordinates (see Problem 12 of Problem Set 3), compute the distance between (0,0,0) and (2,3,6).

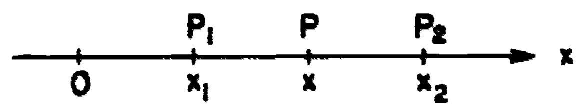
   (b) Write a formula for the distance between (0,0,0) and (x,y,z).

   (c) Write a formula for the distance between $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$.

7. <u>The Mid-Point Formula</u>.

In Section 8 we will be proving geometric theorems by the use of coordinate systems. In some of these proofs, we will need to find the coordinates of the midpoint of a segment P_1P_2 in terms of the coordinates of P_1 and P_2.

First let us take the case where P_1 and P_2 are on the x-axis, with $x_1 < x_2$, like this:



and P is the midpoint, with coordinate x. Since $x_1 < x < x_2$, we know that $P_1P = x - x_1$ and $PP_2 = x_2 - x$.

Since P is the midpoint, this gives

$$x - x_1 = x_2 - x,$$

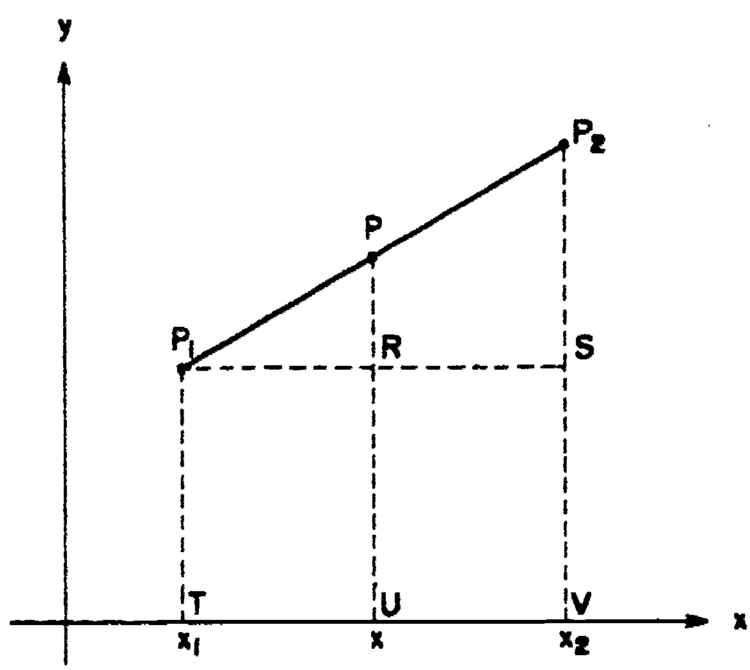
or

$$x = \frac{x_1 + x_2}{2}.$$

In the same way, on the y-axis,

$$y = \frac{y_1 + y_2}{2}.$$

Now we can handle the general case easily:



Since P is the midpoint of $\overline{P_1P_2}$ it follows by similar triangles that R is the midpoint of $\overline{P_1S}$. Since opposite sides of a rectangle are congruent, U is the midpoint of $\overline{TV}$. Therefore,

$$x = \frac{x_1 + x_2}{2} .$$

In the same way, projecting into the y-axis, we can show that

$$y = \frac{y_1 + y_2}{2} .$$

Thus, we have proved:

<u>Theorem 5</u>. (The Midpoint Formula.) Let $P_1 = (x_1,y_1)$ and let $P_2 = (x_2,y_2)$. Then the midpoint of $\overline{P_1P_2}$ is the point

$$P = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}).$$

Problem Set 5.

1. Visualize the points whose coordinates are listed below and compute mentally the coordinates of the midpoint of the segment joining them.

   (a) (0,0) and (0,12).
   (b) (0,0) and (-5,0).
   (c) (1,0) and (3,0).
   (d) (0,-7) and (0,7).
   (e) (4,4) and (-4,-4).

2. Use the midpoint formula to compute the coordinates of the midpoint of the segments joining points with the following coordinates.

   (a) (5,7) and (11,17).
   (b) (-9,3) and (-2,-6).
   (c) $(\frac{1}{2},\frac{1}{5})$ and $(\frac{1}{3},\frac{1}{8})$.
   (d) (2.51,-1.33) and (0.65,3.55).
   (e) (a,0) and (b,c).
   (f) (r + s,r - s) and (-r,s).

3. (a) One endpoint of a segment is (4,0); the midpoint is (4,1). Visualize the location of these points and state, without applying formulas, the coordinates of the other endpoints.

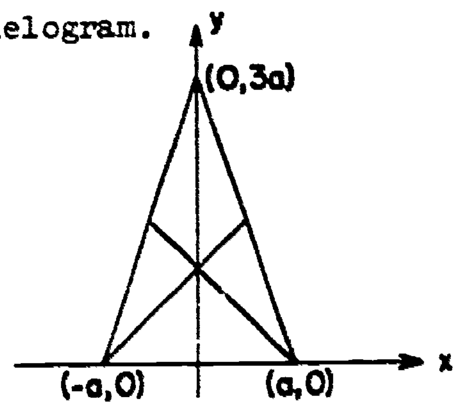
(b) One endpoint of a segment is (13,19). The midpoint is (-9,30). Compute the x and y coordinates of the other endpoint by the appropriate formulas.

4. A quadrilateral is a square if its diagonals are congruent, perpendicular, and bisect each other. Show this to be the case for the quadrilateral having vertices, A(2.1), B(7,4). C(4,9), and D(-1,6).

5. If the vertices of a triangle are A(5,-1), B(1,5) and C(-3,1), what are the lengths of its medians?

6. Given the quadrilateral joining A(3,-2), B(-3,4), C(1,8) and D(7,4), show that the quadrilateral formed by joining its midpoints in order is a parallelogram.

7. Using coordinates, prove that two of the medians of the triangle with vertices (a,0), (-a,0) and (0,3a) are perpendicular to each other.



8. Relocate point P in the figure preceding Theorem 5, so that $PP_1 = \frac{1}{3}P_1P_2$ and find formulas for the coordinates of P in terms of the coordinates of P_1 and P_2. (P is between P_1 and P_2, and $x_2 > x_1$.)

*9. (a) Prove: If $P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$ and $P = (x,y)$ and if P is between P_1 and P_2 such that $\frac{PP_1}{PP_2} = \frac{r}{s}$, then $x = \frac{rx_2 + sx_1}{r + s}$ and $y = \frac{ry_2 + sy_1}{r + s}$.

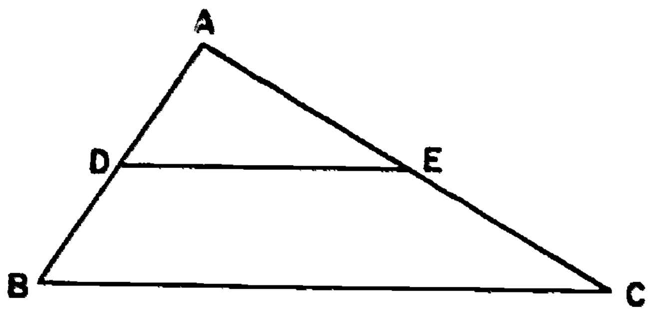
(b) Use the result of part (a) to find a point P on the segment joining $P_1(5,11)$ and $P_2(25,36)$ such that $\frac{PP_1}{PP_2} = \frac{3}{5}$.

8. Proofs of Geometric Theorems.

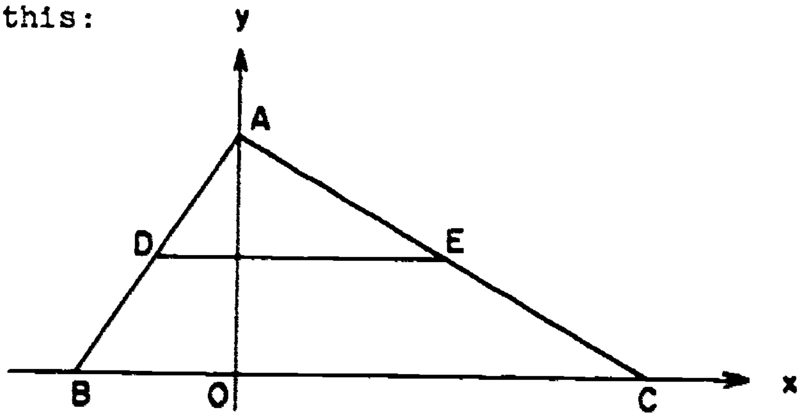
Let us now put our coordinate systems to work in proving a few geometric theorems. We start with a theorem that you may have already proved by other methods.

Theorem A. The segment between the midpoints of two sides of a triangle is parallel to the third side and half as long.

Restatement: In $\triangle ABC$ let D and E be the midpoints of $\overline{AB}$ and $\overline{AC}$. Then $\overline{DE} \| \overline{BC}$ and $DE = \frac{1}{2}BC$.



Proof: The first step in using coordinates to prove a theorem like this is to introduce a suitable coordinate system. That is, we must decide which line is to be the x-axis, which the y-axis, and which direction to take as positive along each axis. We have many choices, and sometimes a clever choice can greatly simplify our work. In the present case it seems reasonably simple to take $\overleftrightarrow{BC}$ as our x-axis, with $\overrightarrow{BC}$ as the positive direction. The y-axis we take to pass through A, with $\overrightarrow{OA}$ as the positive direction, like this:



The next step is to determine the coordinates of the various points of the figure. The x-coordinate of A is zero; the y-coordinate could be any positive number, so we write $A = (0,p)$, with the only restriction on p being $p > 0$. Similarly, $B = (q,0)$ and $C = (r,0)$, with $r > q$. (Note that we might have any of the cases $q < r < 0$, $q < r = 0$, $q < 0 < r$, $0 = q < r$, $0 < q < r$. Our figure illustrates the third case.) The coordinates of D and E can now be found by the midpoint formula. We get

$$D = (\tfrac{q}{2},\tfrac{p}{2}), \quad E = (\tfrac{r}{2},\tfrac{p}{2}).$$

Therefore, the slope of $\overleftrightarrow{DE}$ is

$$\frac{\frac{p}{2} - \frac{p}{2}}{\frac{r}{2} - \frac{q}{2}} = \frac{0}{\frac{r - q}{2}} = 0,$$

(since $q \neq r$, the denominator is not zero).
Likewise, the slope of $\overleftrightarrow{BC}$ is

$$\frac{0 - 0}{\frac{r}{2} - \frac{q}{2}} = 0;$$

and so $\overline{DE} \| \overline{BC}$. Finally, by the distance formula,

$$DE = \sqrt{(\tfrac{r}{2} - \tfrac{q}{2})^2 + (\tfrac{p}{2} - \tfrac{p}{2})^2} = \frac{r - q}{2},$$

and

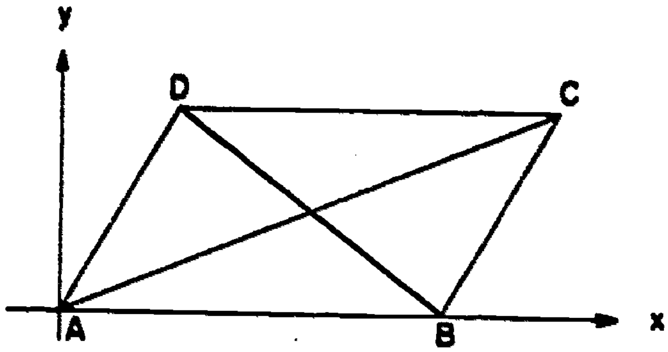
$$BC = \sqrt{(r - q)^2 + (0 - 0)^2} = r - q,$$

so that $DE = \frac{1}{2}BC$.

The algebra in this proof can be made even easier by a simple device. Instead of setting $A = (0,p)$, $B = (q,0)$, $C = (r,0)$ we could just as well have put $A = (0,2p)$, $B = (2q,0)$, $C = (2r,0)$; that is, take p, q and r as half the coordinates of the points A, B and C. If we do it this way, then no fractions arise when we divide by 2 in the midpoint formula. This sort of thing happens fairly often; foresight at the beginning can take the place of patience later on.

<u>Theorem B</u>. If the diagonals of a parallelogram are congruent, the parallelogram is a rectangle.

Restatement: Let ABCD be a parallelogram, and let $AC = BD$. Then ABCD is a rectangle.



Proof: Let us take the axes as shown in the figure. Then $A = (0,0)$, and $B = (p,0)$ with $p > 0$. If we assume nothing about the figure except that ABCD is a parallelogram D could be anywhere in the upper half-plane, so that $D = (q,r)$ with $r > 0$, but no other restriction on q or r. However, C is now determined by the fact that ABCD is a parallelogram. It is fairly obvious (see the preceding proof for details) that for $\overline{DC}$ to be parallel to $\overline{AB}$ we must have $C = (s,r)$. s can be determined by the condition $\overline{BC} \| \overline{AD}$, like this:

$$\text{slope of } \overline{BC} = \text{slope of } \overline{AD},$$

$$\frac{r - 0}{s - p} = \frac{r - 0}{q - 0}, \quad \text{or} \quad \frac{r}{s - p} = \frac{r}{q},$$

$$rq = r(s - p),$$

$$q = s - p, \quad (\text{since } r \neq 0)$$

$$s = p + q.$$

(The coordinates $(p + q, r)$ for C can be written down by inspection if one is willing to assume earlier theorems about parallelograms, for example, that ABCD is a parallelogram if $\overline{AB} \| \overline{CD}$ and $AB = CD$.)

Now we finally put in the condition that $AC = BD$. Using the distance formula, we get

$$\sqrt{(p+q-0)^2 + (r-0)^2} = \sqrt{(q-p)^2 + (r-0)^2}.$$

Squaring gives

$$(p + q)^2 + r^2 = (q - p)^2 + r^2,$$

$$p^2 + 2pq + q^2 + r^2 = q^2 - 2pq + p^2 + r^2,$$

or

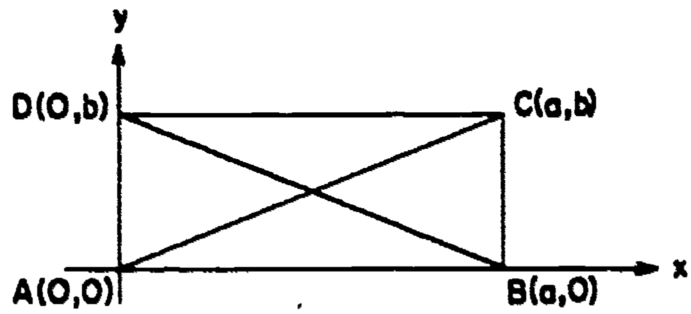
$$4pq = 0.$$

Now $4 \neq 0$ and $p \neq 0$; hence, $q = 0$. This means that D lies on the y-axis, so that $\angle BAD$ is a right angle and ABCD is a rectangle.

Problem Set 6.

Prove the following theorems using coordinate geometry:

1. The diagonals of a rectangle have equal lengths. (Hint: Place the axes as shown.)

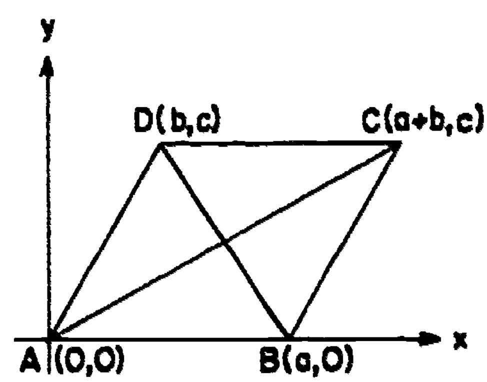


2. The midpoint of the hypotenuse of a right triangle is equidistant from its three vertices.

3. Every point on the perpendicular bisector of a segment is equidistant from the ends of the segment. (Hint: Select the axes in a position which will make the algebraic computation as simple as possible.)

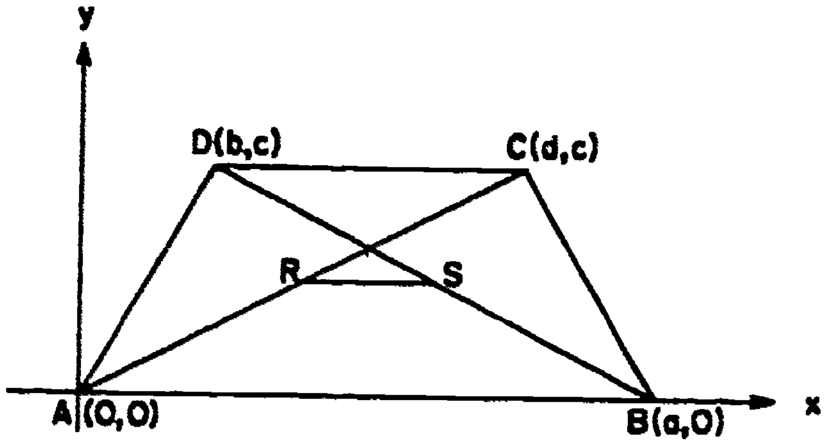
4. Every point equidistant from the ends of a segment lies on the perpendicular bisector of the segment.

5. The diagonals of a parallelogram bisect each other. (Hint: Give the vertices of parallelogram ABCD the coordinates shown in the diagram. Show that both diagonals have the same midpoint.)

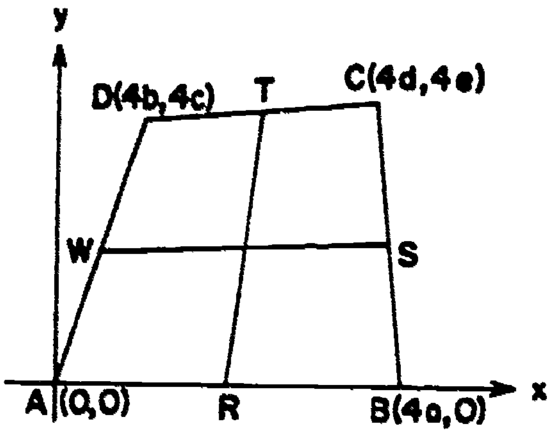


6. The line segment joining the mid-points of the diagonals of a trapezoid is parallel to the bases and equal in length to half the difference of their lengths.

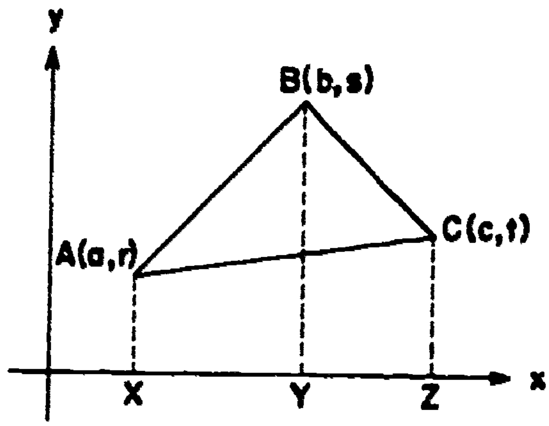
In the figure R and S are midpoints of the diagonals $\overline{AC}$ and $\overline{BD}$ of trapezoid ABCD.



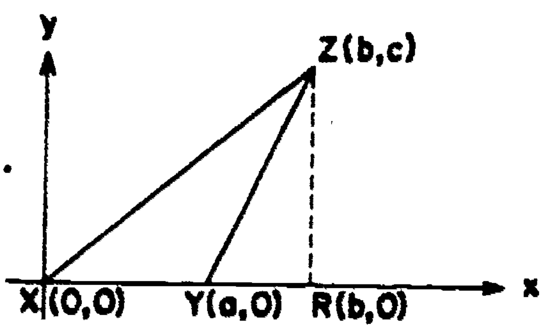
7. The segments joining mid-points of opposite sides of any quadrilateral bisect each other. (The 4's in the diagram are suggested by the fact the midpoints of segments joining mid-points must be found.)



8. The area of $\triangle ABC$ is $\dfrac{a(t-s)+b(r-t)+c(s-r)}{2}$, where $A = (a,r)$, $B = (b,s)$ and $C = (c,t)$. (Hint: Find three trapezoids in the figure.)



9. Given: In $\triangle XYZ$, $\angle X$ is acute and $\overline{ZR}$ is an altitude.
Prove: $ZY^2 = XZ^2 + XY^2 - 2XY \cdot XR$.



10. If ABCD is any quadrilateral with diagonals $\overline{AC}$ and $\overline{BD}$, and if M and N are the midpoints of these diagonals, then $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$.

11. In $\triangle ABC$, $\overline{CM}$ is a median to side $\overline{AB}$. Prove $AC^2 + BC^2 = \frac{AB^2}{2} + 2MC^2$.

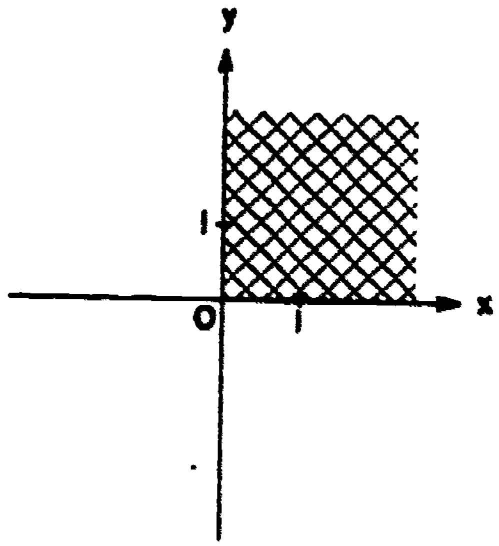
---

9. <u>The Graph of a Condition</u>.

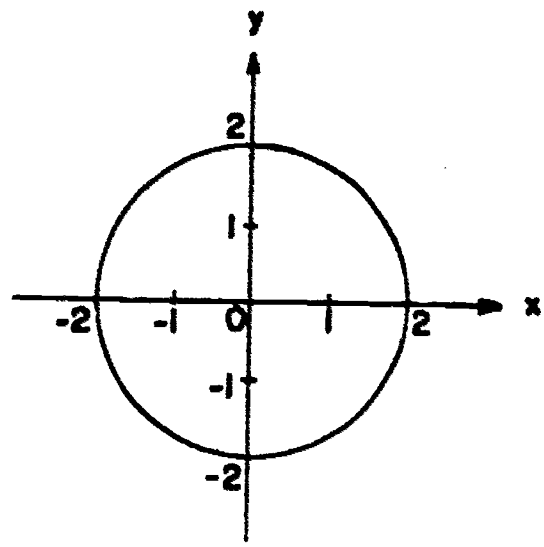
By a <u>graph</u> we mean simply a figure in the plane; that is, a set of points. For example, triangles, rays, lines and half-planes are graphs. We can describe a graph by stating a condition which is satisfied by all points of the graph, and by no other points. Here are some examples showing a condition, a description of the graph, and the figure for each:

Condition	Graph
1. Both of the coordinates of the point P are positive.	1. The first quadrant.
2. The distance OP is 2.	2. The circle with center at the origin, and radius 2.
3. $OP < 1$	3. The interior of the circle with center at the origin and radius 1.
4. $x = 0$.	4. The y-axis.
5. $y = 0$.	5. The x-axis.
6. $x \geq 0$ and $y = 0$.	6. The ray $\overrightarrow{OA}$, where $A = (1,0)$.
7. $x = 0$ and $y \leq 0$.	7. The ray $\overrightarrow{OB}$, where $B = (0,-1)$.

The seven graphs look like this:



1.



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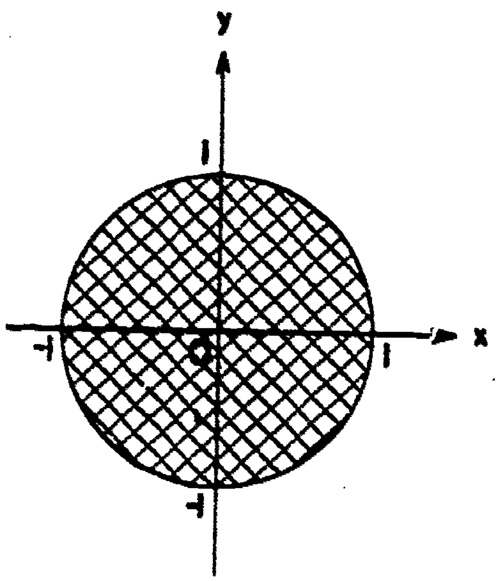

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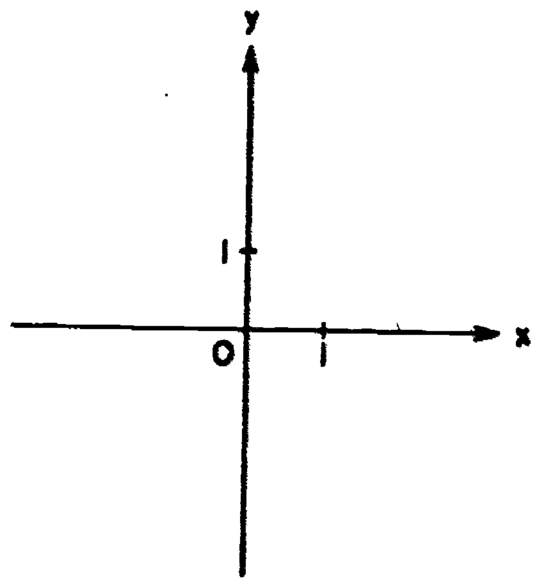

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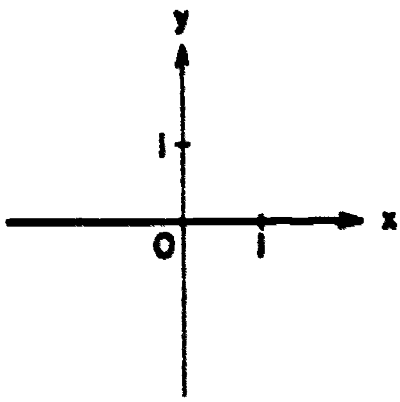

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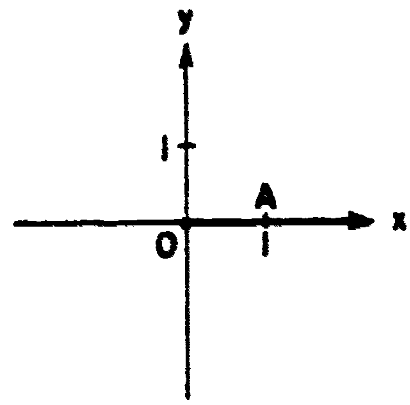

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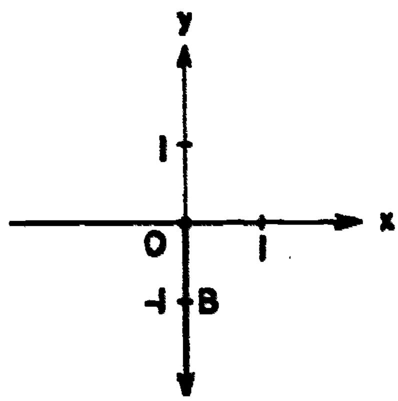

7.

You should check carefully, in each of these cases, that the graph is really accurately described by the condition in the left-hand column above. Notice that we use diagonal cross-hatching to indicate a region.

If a graph is described by a certain condition, then the graph is called the graph of that condition. For example, the first quadrant is the graph of the condition $x > 0$ and $y > 0$; the circle in Figure 2 is the graph of the condition $OP = 2$; the y-axis is the graph of the condition $x = 0$; the x-axis is the graph of the condition $y = 0$; and so on.

Very often the condition describing a graph will be stated in the form of an equation. In these cases we naturally speak of the graph of the given equation.

Problem Set 7

Sketch and describe the graphs of the conditions stated below:

1. (a) $x = 5$. (b) $|x| = 5$.

2. (a) $y > 3$. (b) $|y| < 3$.

3. $0 < x < 2$.

4. $-1 \leq x \leq 5$.

5. $-2 \leq y < 2$.

6. $x < 0$ and $y > 0$.

7. $x > 3$ and $y < -1$.

8. (a) x is a positive integer.
   (b) y is a positive integer.
   (c) Both x and y are positive integers.

9. $x > 0$, $y > 0$, and $y > x$.

10. $1 \leq x < 3$ and $1 \leq y \leq 5$.

*11. $|x| < 4$ and $|y| < 4$.

*12. $|x| < 4$ and $|y| = 4$.

*13. $y = |x|$.

14. $|x| = |y|$.

*15. $|x| + |y| = 5$.

*16. Make a table of some number pairs which satisfy the following sentences. Use these to sketch the graph of each.

(a) $2x + y - 1 = 0$
(b) $y = x^2$
(c) $y - x^2 = 2$
(d) $(x - 1)y = 0$
(e) $xy + 3x = 0$
(f) $y = 2x^2 - x$
(g) $x = |y - 2|$
(h) $y \leq x + 3$
(i) $x > y^2$
(j) $y > |x|$

---

10. How to Describe a Line by an Equation.

We are going to show that any line is the graph of a simple type of equation. We start by considering the condition which characterizes the line.

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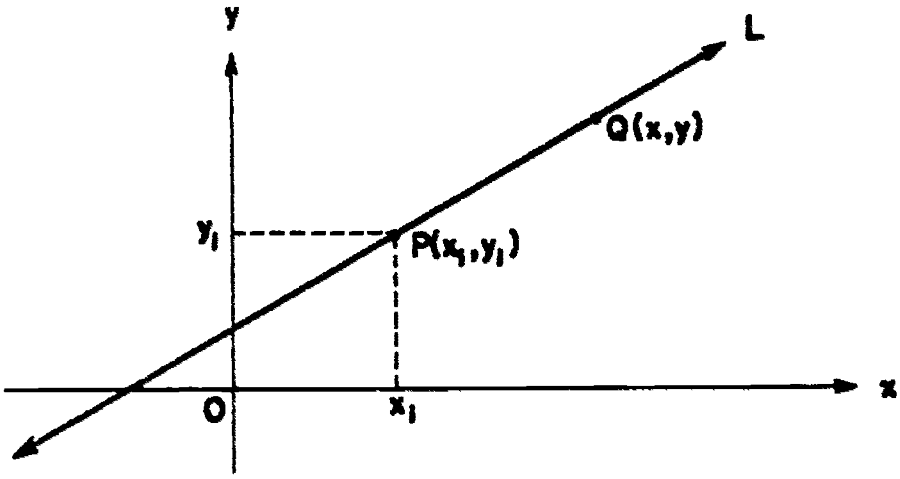
Consider a non-vertical line L, with slope m. Let P be a point of L, with coordinates (x_1, y_1).



Suppose that Q is some other point of L, with coordinates (x,y). Since $\overline{PQ}$ lies in L the slope of $\overline{PQ}$ must be m, and the coordinates of Q must satisfy the condition

$$\frac{y - y_1}{x - x_1} = m.$$

Notice that __this__ equation is __not__ satisfied by the coordinates of the point P, because when $x = x_1$ and $y = y_1$, the left-hand side of the equation becomes the nonsensical expression $\frac{0}{0}$, which is not equal to m (or to anything else, for that matter). If we multiply both sides of this equation by $x - x_1$ with $x \neq x_1$, we get

$$y - y_1 = m(x - x_1).$$

This equation is still satisfied for every point on the line different from P. And it is also satisfied for the point P itself, because when $x = x_1$ and $y = y_1$, the equation takes the form $0 = 0$, which is a true statement.

This is summarized in the following theorem:

__Theorem 6__. Let L be a non-vertical line with slope m, and let P be a point of L, with coordinates (x_1, y_1). For every point $Q = (x,y)$ of L, the equation $y - y_1 = m(x - x_1)$ is satisfied.

4

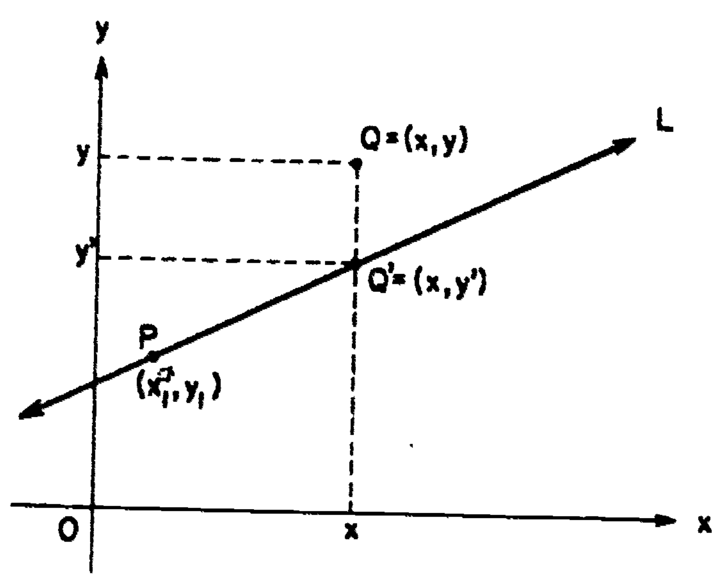
You might think at first that we have proved that the line L is the graph of the equation $y - y_1 = m(x - x_1)$. But to know that the latter is true we need to know that

(1) Every point on L satisfies the equation;

(2) Every point that satisfies the equation is on L.

We have only shown (1), so we have still to show (2). We shall do this indirectly, by showing that if a point is not on L, then it does not satisfy the equation.

Suppose that $Q = (x,y)$ is not on L. Then there is a point $Q' = (x,y')$ which is on L, with $y' \neq y$, like this:



By Theorem 1,
$$\frac{y' - y_1}{x - x_1} = m;$$

hence,
$$y' = y_1 + m(x - x_1).$$

Since $y' \neq y$, this means that
$$y \neq y_1 + m(x - x_1).$$

Therefore,
$$y - y_1 \neq m(x - x_1).$$

Therefore, the equation is satisfied only by points of the line.

We have now proved the very important theorem:

Theorem 7. The graph of the equation
$$y - y_1 = m(x - x_1)$$
is the line that passes through the point (x_1,y_1) and has slope m.

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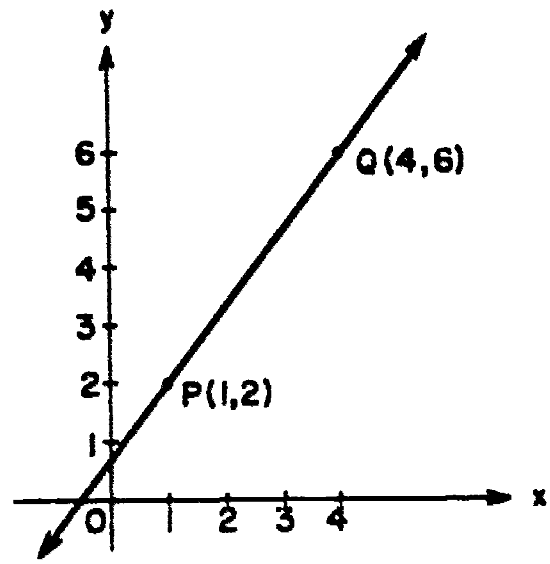
The equation given in Theorem 7 is called the <u>point-slope form</u> of the equation of the line. Let us take an example:



Here we have a line that passes through the points $P = (1,2)$ and $Q = (4,6)$. The slope is

$$m = \frac{6-2}{4-1} = \frac{4}{3}.$$

Using $P = (1,2)$ as the fixed point, we get the equation

$$(1) \qquad y - 2 = \frac{4}{3}(x - 1).$$

(Here $y_1 = 2$, $x_1 = 1$, and $m = \frac{4}{3}$.) In an equivalent form, this becomes

$$(2) \qquad 3y - 6 = 4x - 4, \qquad \text{(How?)}$$

or

$$(3) \qquad 4x - 3y = -2.$$

Notice, however, that while Equation (3) is simpler to look at if all we want to do is look at it, the Equation (1) is easier to interpret geometrically. Theorem 7 tells us that the graph of Equation (1) is the line that passes through the point $P = (1,2)$ and has slope $\frac{4}{3}$.

The student can readily verify that we will get the same or an equivalent equation if we use Q as the fixed point instead of P.

Given an equation in the point-slope form, it is easy to see what the line is. For example, suppose that we have given the equation

$$y - 2 = 3(x - 4).$$

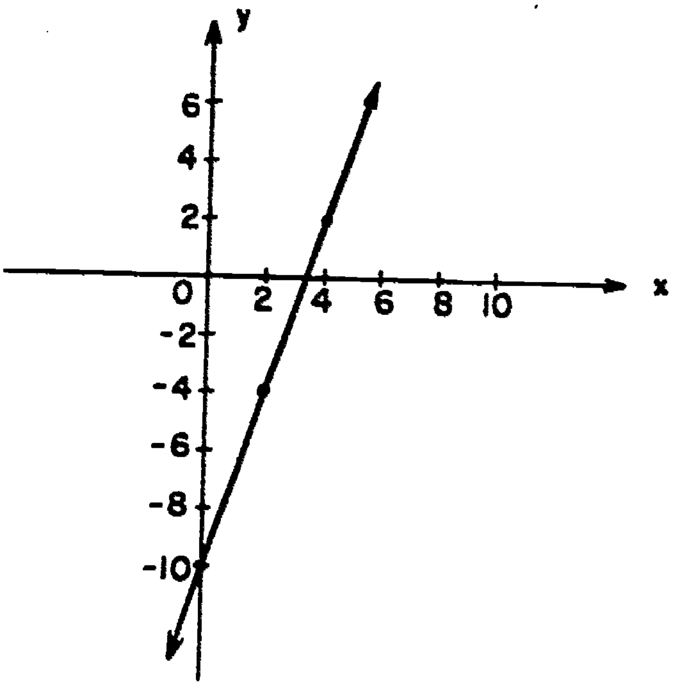
The line contains the point $(4,2)$, and has slope $m = 3$. To draw a line on graph paper, we merely need to know the coordinates of one more point. If $x = 0$, then

$$y - 2 = -12$$

and

$$y = -10.$$

Therefore, the point $(0,-10)$ is on the line, and we can complete the graph:



Logically speaking, this is all that we need. As a practical matter, it is a very good idea to check the coordinates of one more point. This point can be selected anywhere along the line, but to serve as a good check it should not be too near the other two points. If we take $x = 2$, we get

$$y - 2 = -6, \quad \text{or} \quad y = -4.$$

As well as we can judge from the figure, the point $(2,-4)$ lies on the line.

At the beginning of this section we promised to show that any line is the graph of a simple type of equation. We have shown this for any non-vertical line, but we must still consider a vertical line. Suppose a vertical line crosses the x-axis at the point with coordinates $(a,0)$, as in the figure.

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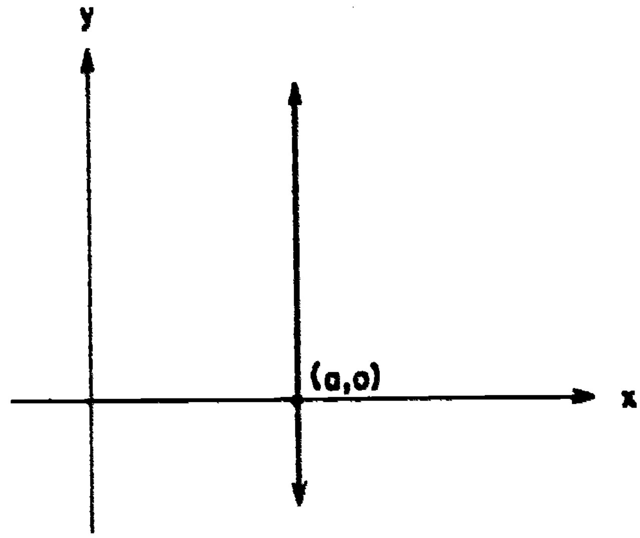


Since the vertical line is perpendicular to the x-axis, every point of the line has its x-coordinate equal to a. Furthermore, any point not on the line will have its x-coordinate not equal to a. Hence, the condition which characterizes the vertical line is $x = a$, certainly a very simple type of equation.

Problem Set 8.

In each of the following problems, we have given the coordinates of a point P and the value of the slope m. Write the point-slope form of the equation of the corresponding line, and draw the graph. Check your work by checking the coordinates of at least one point that was not used in plotting the line. It is all right to draw several of these graphs on the same set of axes, as long as the figures do not become too crowded.

1. $P = (-1,2)$, $m = 4$.

2. $P = (1,-1)$, $m = -1$.

3. $P = (0,5)$, $m = -\frac{1}{3}$.

4. $P = (-1,-4)$, $m = \frac{5}{2}$.

5. $P = (3,-2)$, $m = 0$.

By changing to a point-slope form where necessary, show that the graph of each of the following equations is a line. Then draw the graph and check, as in the preceding problems.

6. $y - 1 = 2(x - 4)$.

7. $y = 2x - 7$.

8. $2x - y - 7 = 0$.

9. $y + 5 = \frac{1}{3}(x + 3)$.

10. $x - 3y = 12$.

11. $y = x$.

12. $y = 2x$.

13. $y = 2x - 6$.

14. $y = 2x + 5$.

15. $x = 4$.

16. $x = 0$.

17. $y = 0$.

18. Thinking in three-dimensional coordinates, describe in words the set of points represented by the following equations. For example, $y = 0$ is the equation of the xz-plane; that is, the plane determined by the x and z-axes. (Refer to Problem 12 of Problem Set 3.)

(a) $x = 0$.
(b) $z = 0$.
(c) $x = 1$.
(d) $y = 2$.

---

11. Various Forms of the Equation of a Line.

We already know how to write an equation for a non-vertical line if we know the slope m and the coordinates (x_1, y_1) of one point of the line. In this case we know that the line is the graph of the equation

$$y - y_1 = m(x - x_1),$$

in the point-slope form.

Definition: The point where the line crosses the y-axis is called the y-intercept. If this is the point $(0, b)$, then the point-slope equation takes the form

$$y - b = m(x - 0),$$
$$y = mx + b.$$

This is called the slope-intercept form. The number b is also called the y-intercept of the line. (When we see the phrase y-intercept, we will have to tell from the context whether the number b or the point $(0,b)$ is meant.) Thus, we have the following theorem:

Theorem 8. The graph of the equation

$$y = mx + b$$

is the line with slope m and y-intercept b.

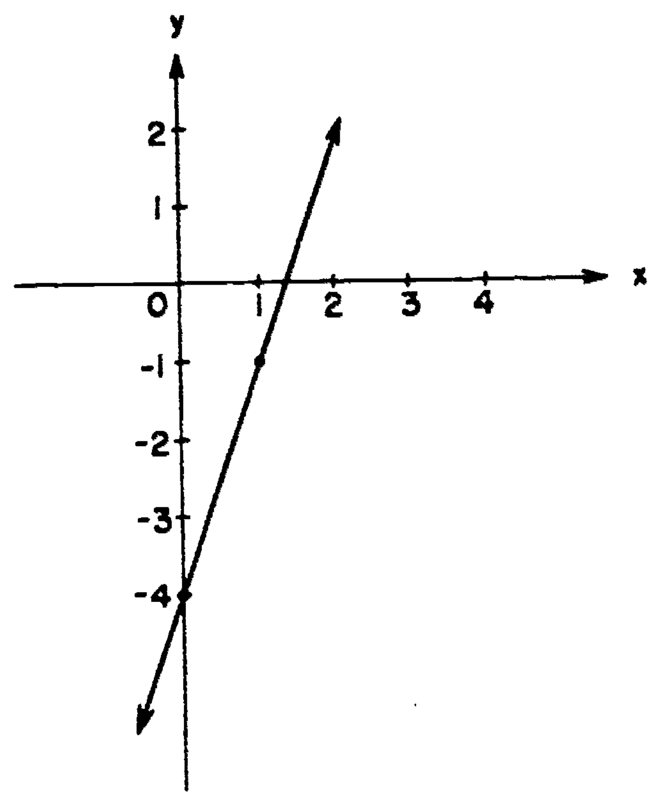
If we have an equation given in this form, then it is easy to draw the graph. All we need to do is to give x any value other than 0, and find the corresponding value of y. We then have the coordinates of two points on the line, and can draw the line. For example, suppose that we have given

$$y = 3x - 4.$$

Obviously, the point $(0,-4)$ is on the graph. Setting $x = 2$, we get

$$y = 6 - 4 = 2.$$

Therefore, the point $(2,2)$ is on the line, and the line therefore looks like this:



As a check, we find that for $x = 1$,

$$y = 3 - 4 = -1,$$

and the point $(1,-1)$ lies on the graph, as well as we can judge.

Notice that once we have Theorem 8, we can prove that certain equations represent lines, by converting them to the slope-intercept form. For example, suppose we have given

$$(1) \qquad 3x + 2y + 4 = 0.$$

This is algebraically equivalent to the equation

$$2y = -3x - 4,$$

or

$$(2) \qquad y = -\frac{3}{2}x - 2.$$

Being equivalent, Equations (1) and (2) have the same graph. The graph of (2) is a line; namely, the line with slope $m = -\frac{3}{2}$ and y-intercept $b = -2$. The graph of (1) is the same line.

---

12. The General Form of the Equation of a Line.

Theorem 8, of course, applies only to non-vertical lines, because these are the ones that have slopes. Vertical lines are very simple objects, algebraically speaking, because they are the graphs of simple equations, of the form

$$x = a.$$

Thus, we have two kinds of equations ($y = mx + b$ and $x = a$) for non-vertical and vertical lines, respectively. We can tie all this together, including both cases, in the following way.

Definition: By a linear equation in x and y we mean an equation of the form

$$Ax + By + C = 0,$$

where A and B are not both zero.

The following two theorems describe the relation between geometry and algebra, as far as lines are concerned:

Theorem 9. Every line in the plane is the graph of a linear equation in x and y.

Theorem 10. The graph of a linear equation in x and y is a line.

Now that we have got this far, both of these theorems are very easy to prove.

Proof of Theorem 9: Let L be a line in the plane. If L is vertical, then L is the graph of an equation

$$x = a,$$

or

$$x - a = 0.$$

This has the form $Ax + By + C = 0$, where $A = 1$, $B = 0$, $C = -a$. A and B are not both zero, because $A = 1$, and so the equation is linear.

If L is not vertical, then L has a slope m and crosses the y-axis at some point $(0,b)$. Therefore, L is the graph of the equation

$$y = mx + b,$$

or

$$mx - y + b = 0.$$

This has the form $Ax + By + C = 0$, where $A = m$, $B = -1$, $C = b$. A and B are not both zero, because $B = -1$. Therefore, the equation is linear. (Notice that it can easily happen that $m = 0$; this holds true for all horizontal lines. Notice also that the equation is not unique: e.g., $2Ax + 2By + 2C = 0$ has the same graph as $Ax + By + C = 0$.)

Proof of Theorem 10: Given the equation $Ax + By + C = 0$ with A and B not both zero.

Case 1. If $B = 0$, then the equation has the form

$$Ax = -C.$$

Since $B = 0$, we know that $A \neq 0$. Therefore, we can divide by A, getting

$$x = -\frac{C}{A}.$$

The graph of this equation is a vertical line.

Case 2. Suppose that $B \neq 0$. Then we can divide by B, getting

$$\frac{A}{B}x + y + \frac{C}{B} = 0,$$

or

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

The graph of this equation is a line; namely, the line with slope $m = -\frac{A}{B}$ and y-intercept $b = -\frac{C}{B}$.

To make sure that you understand what has been proved, in Theorems 9 and 10, you should notice carefully a certain thing that has <u>not</u> been proved. We have <u>not</u> proved that if a given equation has a line as its graph, then the equation is linear. And, in fact, this latter statement is not true. For example, consider the equation

$$x^2 = 0.$$

Now the only number whose square is zero is the number zero itself. Therefore, the equation $x^2 = 0$ says the same thing as the equation $x = 0$. Therefore, the graph of the equation $x^2 = 0$ is the y-axis, which is, of course, a line. Similarly, the graph of the equation

$$y^{17} = 0$$

is the x-axis.

The same sort of thing can happen in cases where it is not so easy to see what is going on. For example, take the equation

$$x^2 + y^2 = 2xy.$$

This can be written in the form

$$x^2 - 2xy + y^2 = 0,$$

or

$$(x - y)^2 = 0.$$

The graph is the same as the graph of the equation

$$x - y = 0,$$

or

$$y = x.$$

The graph is a line.

Notice that the proof of Theorem 10 gives us a practical procedure for getting information about the line from the general equation. If $B = 0$, then we have the vertical line given by the equation

$$x = -\frac{C}{A}.$$

Otherwise, we solve for y, getting

$$y = -\frac{A}{B}x - \frac{C}{B},$$

where the slope is $m = -\frac{A}{B}$

and the y-intercept is $b = -\frac{C}{B}.$

Problem Set 9.

Sketch the graphs of the following equations:

1. $2x + 5y = 7$.
2. $\frac{1}{2}y - 2x + 3 = 0$.
3. $x + 4 = 0$.
4. $y + 4 = 0$.

Describe the graphs of the following equations:

5. $0 \cdot x + 0 \cdot y = 0$.
6. $0 \cdot x + 0 \cdot y = 2$.
7. $x^2 + y^2 = 0$.
8. $x^2 = -1$.

Sketch the graphs of the following conditions:

9. $3x + 4y = 0$ and $x \leq 0$.
10. $5x - 2y = 0$ and $5 \leq y \leq 10$.
11. $(x + y)^2 = 0$.
12. $(y - 1)^{54} = 0$.

Find linear equations $(Ax + By + C = 0)$ of which the following lines are the graphs. State the values for A, B, C in your answer.

13. The line through $(1,2)$ with slope 3.
14. The line through $(1,0)$ and $(0,1)$.
15. The line with slope 2 and y-intercept -4.
16. The x-axis.
17. The y-axis.
18. The horizontal line through $(-5,-3)$.
19. The vertical line through $(-5,-3)$.
20. The line through the origin and the midpoint of the segment with endpoints $(3,2)$ and $(7,0)$.

---

13. Intersection of Lines.

Suppose that we have given the equations of two lines, like this:

$$L_1: \quad 2x + y = 4,$$

$$L_2: \quad x - y = -1.$$

These lines are not parallel because the slope of the first is $m_1 = -2$, and the slope of the second is $m_2 = 1$. Therefore, they intersect in some point $P = (x,y)$. The pair of numbers (x,y) must satisfy both equations. Therefore, the geometric problem of finding the point P is equivalent to the algebraic problem of solving a system of two linear equations in two unknowns.

To solve the system is easy. Adding the two equations, we get

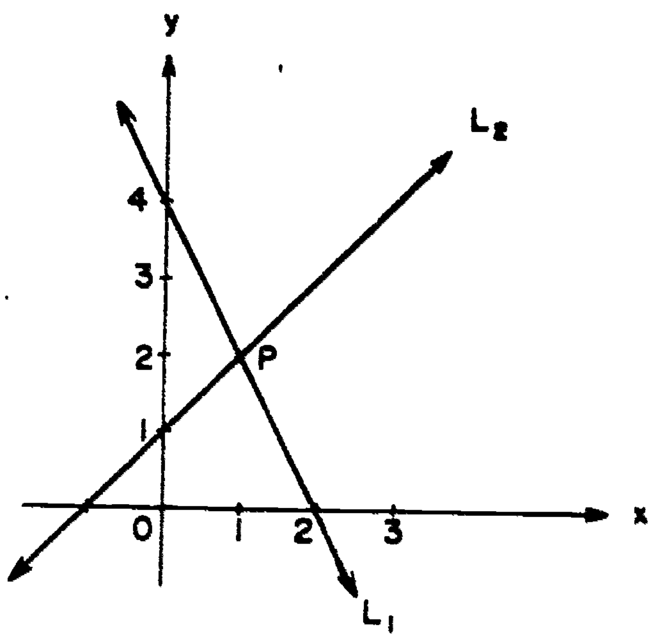
$$3x = 3,$$

or

$$x = 1.$$

Substituting 1 for x in the second equation, we get $y = 2$. The values $x = 1$, $y = 2$ will also satisfy the first equation.

Therefore, $P = (1,2)$. The graph makes this look plausible:



This method always gives the answer to our problem, whenever our problem has an answer; that is, whenever the graphs of the two equations intersect. If the lines are parallel, then the corresponding system of equations will be inconsistent; that is, the solution of the system will be the empty set. This will be plain enough when we try to solve the system.

Problem Set 10.

1. Find the common solution of the following pairs of equations and draw their graphs.

   (a) $y = 2x$ and $x + y = 7$.

   (b) $y = 2x$ and $y - 2x = 3$.

   (c) $x + y = 3$ and $2y = 6 - 2x$.

2. (a) The graphs of which pairs of the equations listed below would be parallel lines?

   (b) Intersecting but not coincident lines?

   (c) Coincident lines?

   The equations are:

   (1) $y = 3x + 1$. (3) $2y = 6x + 2$.

   (2) $y = 4x + 1$. (4) $y - 3x = 2$.

3. Suppose the unit in our coordinate system is 1 mile. How many miles from the origin is the point where the line $y = \frac{1}{1000}x - 4$ crosses the x-axis?

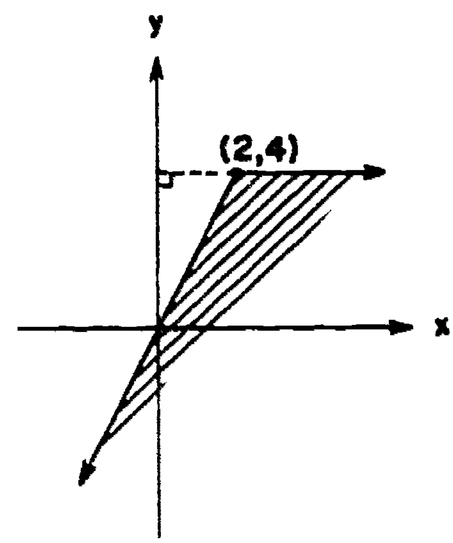
4. Find the intersection of the graphs of the following pairs of conditions:

   (a) $y = 2x$ and $y = 4$.

   (b) $y = 2x$ and $y \geq 4$.

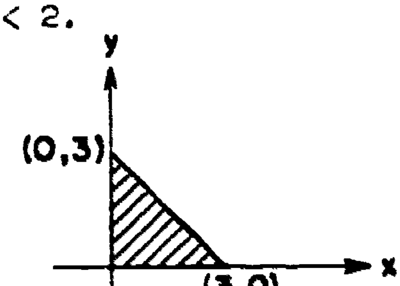
   (c) $y < 2x$ and $y > 4$.

   (d) What pair of conditions will determine the interior of the angle shown in the figure?



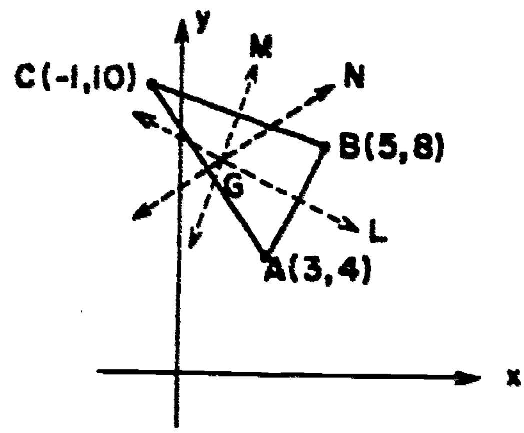
5. (a) Sketch the intersection of the graphs of all three conditions $x + y > 3$, $y < 4$, $x < 2$.

   (b) State the three conditions which would determine the interior of the triangle shown.



6. Find an equation for the perpendicular bisector of the segment with endpoints $(3,4)$ and $(5,8)$.

7. Find equations for the perpendicular bisectors of the sides of $\triangle(3,4)(5,8)(-1,10)$, and show that they intersect in a point.



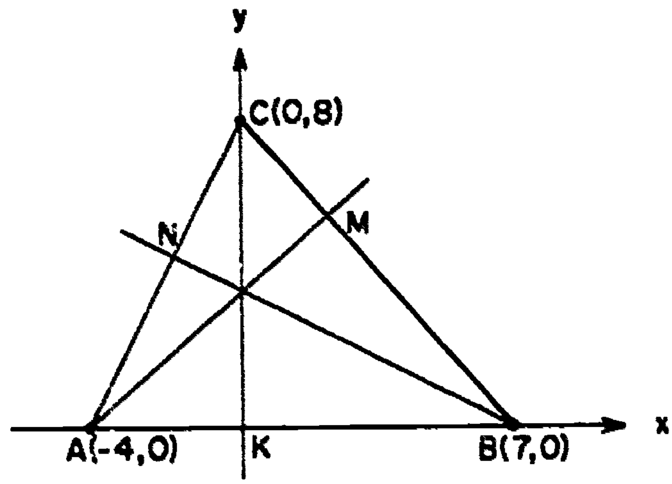
*8. The following instructions were found on an ancient document. "Start from the crossing of King's Road and Queen's Road. Proceeding north on King's Road, find first a pine tree, then a maple. Return to the crossing. West on Queen's Road there is an elm and east on Queen's Road there is a spruce. One magical point is where the elm-pine line meets the maple-spruce line. The other magical point is where the spruce-pine line meets the elm-maple line. The line joining the two magical points meets Queen's Road where the treasure is buried."

A search party found the elm 4 miles from the crossing, the spruce 2 miles from the crossing, and the pine 3 miles from the crossing, but there was no trace of the maple. Nevertheless, they were able to find the treasure from the instructions. Show how this was done.

One man in the party remarked on how fortunate they were to have found the pine still standing. The leader laughed and said, "We didn't need the pine tree either." Show that he was right.

*9. One of the altitudes of the $\triangle ABC$, where $A = (-4,0)$, $B = (7,0)$, $C = (0,8)$, is the y-axis. Why? Prove, using coordinate methods, that the altitudes from A and B meet on that axis. (Hint: Find the intersections of those altitudes with the y-axis.) Do the same for the triangles with vertices $(a,0)$, $(b,0)$, $(0,c)$.

*9. (continued)



*10. The centroid of a triangle is defined as the intersection of the three medians. Prove that the coordinates of the centroid are just the averages of the coordinates of the vertices.

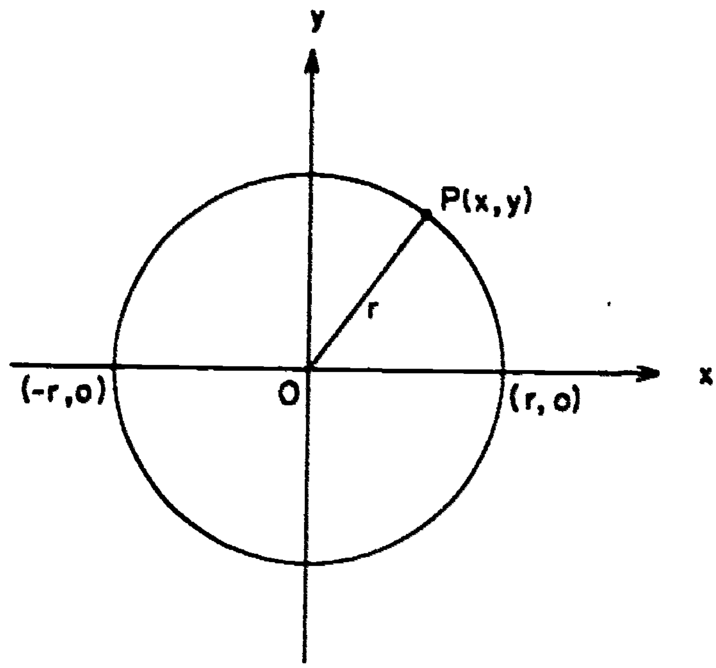
*11. Find the distance from the point $(1,2)$ to the line $x + 3y + 1 = 0$.

*12. Find the distance from the point (a,b) to the line $y = x$.

*13. In the general case of the triangle of Problem 9, let H be the point of concurrence of the altitudes, M the point of concurrence of the medians, and D the point of concurrence of the perpendicular bisectors of the sides. Prove, using Problems 9 and 10 that these three points are collinear, and that M divides $\overline{DH}$ in the ratio two to one (refer to Problem 8 of Problem Set 7).

14. Circles.

Consider the circle with center at the origin and radius P.



This figure is defined by the condition

$$OP = r.$$

Algebraically, in terms of the distance formula, this says that

$$\sqrt{(x - 0)^2 + (y - 0)^2} = r,$$

or

$$x^2 + y^2 = r^2.$$

That is, if $P(x,y)$ is a point of the circle, then $x^2 + y^2 = r^2$. We still have to show that if $x^2 + y^2 = r^2$, then $P(x,y)$ is a point of the circle. This we do by reversing the algebraic steps: If

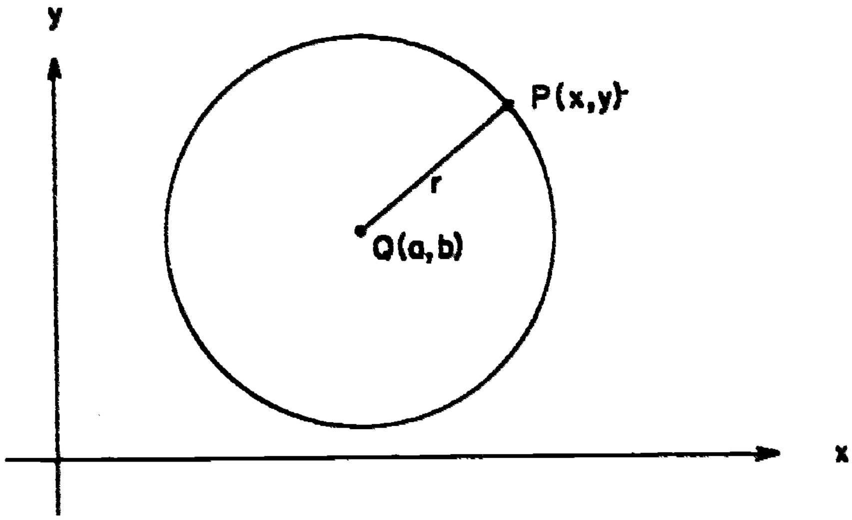
$$x^2 + y^2 = r^2$$

then

$$\sqrt{(x - 0)^2 + (y - 0)^2} = r,$$

since r is a positive number. This equation says that $OP = r$, and so P is a point of the circle.

Consider, more generally, the circle with center at the point $Q = (a,b)$ and radius r.



This is defined by the condition $QP = r$,

or

$$\sqrt{(x - a)^2 + (y - b)^2} = r,$$

or

$$(x - a)^2 + (y - b)^2 = r^2.$$

In this case, also, the algebraic steps can be reversed, and so we can say that

$$(x - a)^2 + (y - b)^2 = r^2$$

is the equation of the circle.

This is the standard form of the equation of the circle, with center (a,b) and radius r. For future reference, let us state this result as a theorem.

Theorem 11. The graph of the equation

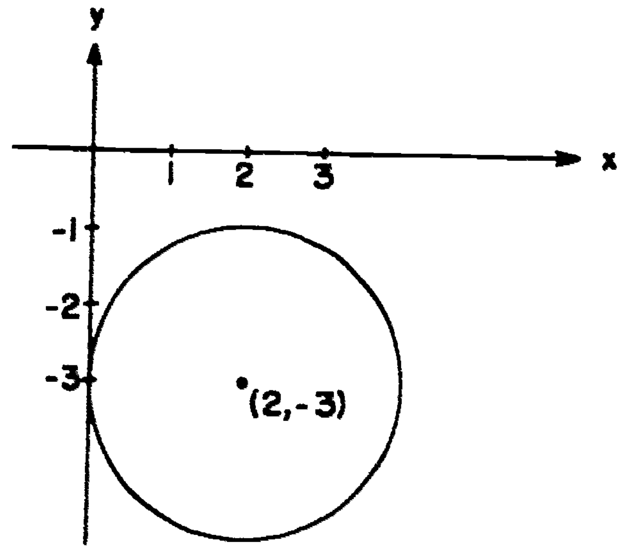
$$(x - a)^2 + (y - b)^2 = r^2$$

is the circle with center at (a,b) and radius r.

If an equation is given in this form, we can read off immediately the radius and the coordinates of the center. For example, suppose that we have given the equation

$$(x - 2)^2 + (y + 3)^2 = 4.$$

The center is the point $(2,-3)$, the radius is 2, and the circle looks like this:



So far, this is easy enough. But suppose that the standard form of the equation has fallen into the hands of someone who likes to "simplify" formulas algebraically. He would have "simplified" the equation like this:

$$x^2 - 4x + 4 + y^2 + 6y + 9 = 4$$
$$x^2 + y^2 - 4x + 6y + 9 = 0.$$

From his final form, it isn't at all easy to see what the graph is. Sometimes we will find equations given in forms like this. Therefore, we need to know how to "unsimplify" these forms so as to get back the standard form

$$(x - a)^2 + (y - b)^2 = r^2.$$

The procedure is this. First we group the terms in x together, and the terms in y together, and write the equation with the constant term on the right, like this:

$$x^2 - 4x \quad + \quad y^2 + 6y \quad = \quad -9.$$

Then we see what constant should be added to the first two terms to complete a perfect square. Recall that to find this constant takes half of the coefficient of x, and square the result. Here we get 4. The same process, applied to the third and fourth terms, shows that we should add 9 in order to make a perfect square. Thus, we are going to add a total of 13 to the left-hand side of the equation. Therefore, we must add 13 to the right-hand side. Now our equation takes the equivalent form

$$x^2 - 4x + 4 + y^2 + 6y + 9 = -9 + 13,$$

or

$$(x - 2)^2 + (y + 3)^2 = 4,$$

as before.

If we multiply out and simplify in the standard form, we get

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0.$$

This has the form

$$x^2 + y^2 + Ax + By + C = 0.$$

Thus, we have the theorem:

<u>Theorem 12</u>. Every circle is the graph of an equation of the form

$$x^2 + y^2 + Ax + By + C = 0.$$

It might seem reasonable to suppose that the converse is also true. That is, we might think that every equation of the form that we have been discussing has a circle as its graph. But this is not true by any means. For example, consider the equation

$$x^2 + y^2 = 0.$$

Here A, B and C are all zero. If x and y satisfy this equation, then x and y are both zero. That is, the graph of the equation is a single point; namely, the origin.

Consider next the equation

$$x^2 + y^2 + 1 = 0.$$

Here $A = B = 0$ and $C = 1$. This equation is not satisfied by the coordinates of any point whatsoever. (Since $x^2 \geq 0$, and $y^2 \geq 0$, and $1 > 0$, it follows that $x^2 + y^2 + 1 > 0$ for every pair of real numbers x and y.) For this equation, the graph has <u>no points at all</u>, and is therefore the empty set.

In fact, the only possibilities are the circle that we would normally expect, plus the two unexpected possibilities that we have just noted.

Theorem 13. Given the equation

$$x^2 + y^2 + Ax + By + C = 0.$$

The graph of this equation is (1) a circle, (2) a point or (3) the empty set.

Proof: Let us complete the square for the terms in x, and complete the square for the terms in y, just as we did in the particular case that we worked out above. This gives

$$x^2 + Ax + \frac{A^2}{4} + y^2 + By + \frac{B^2}{4} = -C + \frac{A^2}{4} + \frac{B^2}{4},$$

or

$$(x + \frac{A}{2})^2 + (y + \frac{B}{2})^2 = \frac{A^2 + B^2 - 4C}{4}.$$

If the fraction on the right is positive, equal to r^2 with $r > 0$, then the graph is a circle with center at $(-\frac{A}{2}, -\frac{B}{2})$ and radius r. If the fraction on the right is zero, then the graph is the single point $(-\frac{A}{2}, -\frac{B}{2})$. If the fraction on the right is negative, then the equation is never satisfied, and the graph contains no points at all.

Problem Set 11.

1. The circle shown has a radius of 5 units. Find the value of:

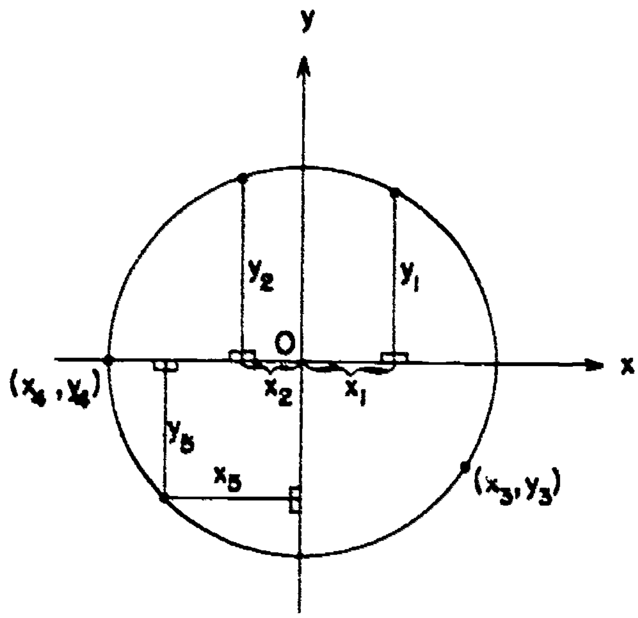
   (a) $x_1^2 + y_1^2$.

   (b) $x_2^2 + y_2^2$.

   (c) $x_3^2 + y_3^2$.

   (d) $x_4^2 + y_4^2$.

   (e) $x_5^2 + y_5^2$.



2. (a) Which of the following eight equations have graphs which are circles?

(b) Which of the circles would have centers at the origin?

(c) Which would have centers on an axis, but not at the origin?

(1) $x^2 + (y - 1)^2 = 9$. (5) $(x - 2)^2 - (y - 9)^2 = 16$.

(2) $y = x^2$. (6) $(x - 2)^2 + (y - 3)^2 = 16$.

(3) $x^2 + y^2 = 7$. (7) $3x^2 + y^2 = 4$.

(4) $1 - x^2 = y^2$. (8) $x^2 + y^2 = 0$.

3. Determine the center and radius of each of the following circles.

(a) $x^2 + y^2 = 3^2$. (f) $(x - 4)^2 + (y - 3)^2 = 36$.

(b) $x^2 + y^2 = 100$. (g) $(x + 1)^2 + (y + 5)^2 = 49$.

(c) $(x - 1)^2 + y^2 = 16$. (h) $x^2 - 2x + 1 + y^2 = 25$.

(d) $x^2 + y^2 = 7$. (i) $x^2 - 2x + y^2 = 24$.

(e) $y^2 = 4 - x^2$. (j) $x^2 + 6x + y^2 - 4y = 12$.

4. A circle has the equation: $x^2 - 10x + y^2 = 0$.

(a) Show algebraically that the points $(0,0)$, $(1,3)$ and $(2,4)$ all lie on the circle.

(b) Find the center and radius of the circle.

(c) Show that if $(1,3)$ is joined to the ends of the diameter on the x-axis, a right angle is formed with vertex at $(1,3)$.

5. (a) Find the points where the circle $(x - 3)^2 + y^2 = 25$ is intersected by the x and y-axes.

(b) Considering portions of the x and y-axes as chords of the circle in part (a), prove that the products of the lengths of the parts into which each chord is divided by the other, are equal.

6. Draw the four circles obtained by choosing the various possible sign combinations in

$$(x \pm 1)^2 + (y \pm 1)^2 = 1.$$

Then write the equations of the circle tangent to all four and containing them. Is there another circle tangent to all four? What is its radius?

7. Draw the 4 circles given by

$$x^2 + y^2 = \pm 10x, \qquad x^2 + y^2 = \pm 10y$$

and write the equation of a circle tangent to all of them.

8. Given the circle $x^2 + y^2 = 16$ and the point $K(-7,0)$.

   (a) Find the equation (in point-slope form) of the line L_m with slope m passing through the point K.

   (b) Find the points (or point) of intersection of L_m and the circle.

   (c) For what values of m is there exactly one point of intersection? Interpret this result geometrically.

9. Find an equation for a circle tangent externally to the circle

$$x^2 + y^2 - 10x - 6y + 30 = 0$$

and also tangent to the x and y-axes.

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15. Sets Satisfying Geometric Conditions.

In Section 9 we considered the question of determining the set of points whose coordinates satisfied some condition. In this section we reverse the question and ask for an algebraic description of the set determined by some geometric condition. The machinery of coordinate geometry is ideally suited for this task. We use the results of the preceding sections to write algebraic descriptions of geometric dontions.

Example 1: Describe the set o. all points at a distance 1 from the origin.

Solution: Geometrically, the circle with center at the origin and radius 1, satisfies this condition. This is a perfectly good description of the set. However, we could still describe the set algebraically by using an equation to express the given geometric condition.

Let $P(x,y)$ by any point satisfying the condition.

Then $$OP = 1.$$

Using the distance formula
$$\sqrt{(x - 0)^2 + (y - 0)^2} = 1$$
or
$$x^2 + y^2 = 1.$$

This algebraic condition is simply a straightforward algebraic translation of the geometric condition.

Example 2: Find the set of all points which are twice as far from the origin as from the point $(2,0)$.

Solution: In this case, we may have no idea what the geometric description of the set is. However, it is still easy to write out the algebraic description.

Suppose $P(x,y)$ is any point of the set and let A be the point $(2,0)$. Then
$$OP = 2(PA)$$
$$\sqrt{x^2 + y^2} = 2\sqrt{(x - 2)^2 + (y - 0)^2}$$
or
$$x^2 + y^2 = 4[(x - 2)^2 + (y - 0)^2].$$

Simplifying, we get $3x^2 + 3y^2 - 16x + 16 = 0$.

From our work in Section 14, this last equation can be shown to be the equation of circle. (What is the center? What is the radius?) Therefore, the set of points containing all those points, and only those points, which are twice as far from the origin as from the point $(2,0)$, is a circle.

Problem Set 12.

In each exercise the graph should be plotted.

1. Write the equation describing the set of points which are at a distance 2 from the origin.

2. Write the equation of the set of all points which are at a distance 1 from the point $C(1,0)$.

3. Write the equation of the set of all points which are at a distance 3 from the point $C(0,2)$.

4. Write the equation of the set of all points which are at a distance 5 from the point $C(2,3)$.

5. Write the equation of the set of all points which are k units from the point $C(-1,3)$.

6. Write the equation of the set of all points at a distance r from the point $C(h,k)$. Describe this set geometrically.

7. Write the equation of the set of all points which are equidistant from the points $A(3,0)$ and $B(5,0)$.

8. Write the equation of the set of all points which are equidistant from the points $A(-2,-5)$ and $P(3,2)$.

*9. Write the equation of the set of all points which are equidistant from the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$. Describe this set geometrically.

10. Write the equation of the set of points each of which is twice as far from $A(-2,0)$ as it is from $B(1,0)$.

11. Write the equation of the set of points each of which is the vertex of a right triangle whose hypotenuse is the line segment joining $(-1,0)$ and $(1,0)$. Describe this set geometrically.

*12. Write the equation of the set of points each of which is the midpoint of a line segment of length 2 having its endpoints on two perpendicular lines.

13. Write the equation of the set of points each of which is the center of a circle which is tangent to the x-axis and which passes through the point $(0,1)$.

14. Write an equation whose only solution is $x = 0$, $y = 0$; that is, give an equation for the origin.

15. Write an equation for the semicircle of radius 2 with center at $(0,0)$ and lying to the left of the y-axis.

16. Write an equation of the set of all points (x,y) such that the area of the triangle with vertices (x,y), $(0,0)$ and $(3,0)$ is 2.

REVIEW PROBLEMS

1. What are the coordinates of the projection into the x-axis of the point $(5,2)$?

2. Three of the vertices of a rectangle are $(-1,-1)$, $(3,-1)$ and $(3,5)$. What is the fourth vertex?

3. An isosceles triangle has vertices $(0,0)$, $(4a,0)$ and $(2a,2b)$. What is the slope of the median from the origin? of the median from $(2a,2b)$?

4. In Problem 3 what is the slope of the altitude which contains the origin?

5. What is the length of each of the medians of the triangle in Problem 3?

6. What is the slope of a line that is parallel to a line which passes through the origin and through $(-2,3)$?

7. The vertices of a quadrilateral are $(0,0)$, $(5,5)$, $(7,1)$ and $(1,7)$. What are the lengths of its diagonals?

8. What are the coordinates of the mid-points of segments joining the pairs of points in Problem 7?

9. The vertices of a square are labeled consecutively, P, Q, R and S. T is the midpoint of $\overline{QR}$ and U is the midpoint of $\overline{RS}$. $\overline{PT}$ intersects $\overline{QU}$ at V.

   (a) Prove that $\overline{PT} \cong \overline{QU}$.

   (b) Prove that $\overline{PT} \perp \overline{QU}$.

   *(c) Prove that $VS = PQ$.

   (Hint: Let $P = (0,0)$ and $Q = (2a,0)$.)

10. Use coordinate geometry to prove the theorem: The median of a trapezoid bisects a diagonal.

11. What is the equation whose graph is the y-axis?

12. A rhombus ABCD has A at the origin and $\overline{AB}$ in the positive x-axis, $m\angle a = 45$, $AB = 6$, C is in the first quadrant. What is the equation of $\overleftrightarrow{AB}$? $\overleftrightarrow{BC}$? $\overleftrightarrow{CD}$?

13. The coordinates of the vertices of a trapezoid are, consecutively, $(0,0)$, $(a,0)$, (b,c) and (d,c). Find the area of the trapezoid in terms of these coordinates.

14. The graphs of the equations $y = \frac{1}{2}x$ and $y = -2x + 5$ are perpendicular to each other at what point?

15. Name the set of points such that the sum of the squares of the distances of each point from the two axes is 4.

16. Write an equation of the circle which has

(a) its radius 7 and center at the origin.
(b) its radius k and center at the origin.
(c) its radius 3 and center at $(1,2)$.

*17. Prove that the line $x + y = 2$ is tangent to the circle $x^2 + y^2 = 2$.
